Policy Research Journal
ISSN (E): 3006-7030 ISSN (P) : 3006-7022 Volume 4, Issue 3, 2026

FRACTIONAL CALCULUS OPERATORS IN GEOMETRIC FUNCTION THEORY AND THEIR SEMIGROUP STRUCTURES

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DOI: https://doi.org/10.5281/zenodo.19399425

Keywords
fractional calculus operators, geometric function theory, univalent functions, bi-univalent functions, Riemann–Liouville operator, Caputo derivative, Erdélyi–Kober integral, operator semigroups, subordination, q-fractional calculus, Fekete–Szegő inequality, radius constants

Article History
Received: 31 January 2026
Accepted: 15 March 2026
Published: 31 March 2026

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Abstract

Fractional calculus operators Riemann–Liouville, Caputo, Erdélyi–Kober, Hadamard, and their q-analogues have emerged as powerful tools in geometric function theory, enabling the study of analytic functions and their subclasses through non-integer order differintegral transforms that capture memory effects, scale-invariant properties, and generalized subordination. This review systematically examines the application of fractional operators to define and characterize new subclasses of univalent, bi-univalent, and multivalent functions in the open unit disk, focusing on their geometric properties (starlikeness, convexity, close-to-convexity, spirallikeness) and subordination relationships. Key developments include fractional integral/differential operators used to construct coefficient bounds, Fekete–Szegő inequalities, radius constants, and inclusion theorems for families such as fractional q-starlike, fractional convex, and fractional close-to-convex functions. The semigroup structures inherent in these operators' closure under composition, identity element, inverses, and algebraic properties are analyzed, revealing that many fractional differintegral operators form continuous semigroups under suitable function spaces and parameter regimes, facilitating the study of evolution equations and operator semigroups in complex analysis. Connections to special functions (q-Fibonacci, q-hypergeometric, Mittag-Leffler) and their convolution operators further enrich the theory, yielding sharp estimates and geometric characterizations. Applications extend to control theory (fractional-order PID/sliding-mode controllers) and signal processing, where semigroup properties ensure stability and robustness. Challenges non-locality, computational complexity, and lack of unified algebraic frameworks are addressed through recent advances in q-fractional calculus and operator theory. The integration of fractional calculus into geometric function theory not only generalizes classical results but also opens avenues for novel subclasses and semigroup-based techniques in analytic function spaces.

1. INTRODUCTION

The exploration of complex analysis through the dual lenses of geometric function theory and fractional calculus has yielded a profound understanding of how non-integer order operators influence the topological and geometric behavior of analytic mappings (Oros & Oros, 2024).

Geometric function theory is primarily concerned with the study of the geometric characteristics of analytic functions, specifically in terms of how they map specific domains, such as the open unit disk, onto other complex structures (Abubaker et al., 2025). The introduction of fractional calculus operators including the Riemann-Liouville, Caputo, and Erdélyi-Kober formulations provides a mechanism to scale and transform these mappings in ways that classical integer-order derivatives cannot (Aldawish & Ibrahim, 2023). This synthesis is further enriched by the inclusion of semigroup structures, which offer a dynamical perspective on the iterative evolution of these functions (Contreras & Díaz-Madrigal, 2005). By viewing the continuous iteration of analytic maps as a one-parameter semigroup, researchers can apply fractional calculus to the infinitesimal generators of these flows, bridging the gap between static geometric properties and temporal functional evolution (Goryainov, 2012).

2. Theoretical Foundations of Geometric Function Theory

Geometric Function Theory (GFT) represents a sophisticated intersection of complex analysis and geometry, where the primary objective is to investigate the mapping properties of analytic functions defined on subsets of the complex plane (Reddy and Murugusundaramoorthy, 2023). The fundamental domain for such investigations is typically the open unit disk, denoted by

$$U = \{z \in \mathbb{C} : | z | < 1\}$$

(Tassaddiq et al., 2024).

The motivation for focusing on the unit disk arises from the Riemann Mapping Theorem, which asserts that any simply connected domain in the complex plane, provided it is not the entire plane itself, can be conformally mapped onto U. This theorem establishes the unit disk as a universal laboratory for studying the properties of holomorphic mappings (Sindhushree et al., 2024).

2.1. The Class of Normalized Analytic Functions

At the heart of Geometric Function Theory (GFT) lies the class $\mathcal{A}$, which consists of functions $f(z)$that are analytic in the unit disk $U$and normalized by the conditions

$$f(0) = 0 \text{and} f'(0) = 1$$

(San and Irmak, 2016). These normalization criteria ensure that the function preserves the origin and maintains a unit derivative at that point, leading to the following Taylor series representation (Breaz et al., 2022):

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

The study of this class is subdivided into various categories based on the geometric nature of the image domain $f(U)$. The most critical subclass is $\mathcal{S}$, the set of all univalent (one-to-one) functions in $\mathcal{A}$(Mehrez, 2023). A function $f$is said to be univalent if it never assumes the same value twice in its domain; that is,

$$f(z_1) = f(z_2) \implies z_1 = z_2$$

Geometrically, this condition ensures that the mapping does not overlap itself, thereby preserving the topology of the domain (Goodman, 1979).

Table 1. Subclasses of Normalized Analytic Functions and Their Analytic Conditions
Important Subclasses in Geometric Function Theory

Subclass	Geometric Characterization	Analytic Condition
$\mathcal{S}^*$	**Starlike:** The domain $f(U)$is starlike with respect to the origin.	$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$
$\mathcal{K}$	**Convex:** The domain $f(U)$is a convex set.	$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$
$\mathcal{C}$	**Close-to-Convex:** The image is close to a convex domain.	$\text{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0$, for some $g \in \mathcal{K}$
$\mathcal{P}$	**Carathéodory Class:** Functions with positive real part in U.	$\text{Re}\,(p(z)) > 0, p(0) = 1$

The relationships between these subclasses are foundational in Geometric Function Theory. For instance, the Alexander Theorem establishes that a function $f$belongs to the class $\mathcal{K}$if and only if

$$zf'(z) \in \mathcal{S}^*.$$

Furthermore, it is a well-known result that every convex function is starlike of order

$$\frac{1}{2}.$$

(Halter-Koch, 2022). The class $\mathcal{P}$serves as a critical analytic tool, as many geometric conditions can be reformulated as requirements that a specific functional of $f$belongs to the class $\mathcal{P}$(Mahmoud et al., 2025).

2.2. Subordination and Extremal Problems

The concept of subordination provides a method for comparing the relative size or range of two analytic functions. We say that a function $f$is subordinate to a function g, denoted by

$$f(z) \prec g(z),$$

if there exists an analytic Schwarz function $w(z)$such that

$$f(z) = g(w(z)).$$

If $g$is univalent, this subordination implies that

$$f(0) = g(0) \text{and} f(U) \subseteq g(U)$$

(Aarthy and Keerthi, 2025).

Subordination theory is essential for defining generalized subclasses, such as those defined by Janowski, which restrict the values of

$$\frac{zf'(z)}{f(z)}$$

to a specific disk in the right half-plane (Liu et al., 2025).

Extremal problems in Geometric Function Theory (GFT) often revolve around finding the maximum or minimum values of coefficients or functionals over a given class. The most famous example is the Bieberbach Conjecture (now de Branges' Theorem), which proves that for $f \in \mathcal{S}$, the coefficients satisfy

$$|\, a_n \,| \leq n, \text{for all } n \geq 2$$

(Choi, 2018).

The Koebe function is given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n\, z^n,$$

and it is the extremal function for this class, mapping the unit disk Uonto the entire complex plane minus a slit along the negative real axis (Pilla, 2021).

3. Fractional Calculus Operators in the Complex Plane

Fractional calculus generalizes the concepts of differentiation and integration to non-integer orders, providing a powerful set of operators for

transforming analytic functions (Beyer, 2025). In Geometric Function Theory, these operators are used to define new subclasses by scaling the Taylor coefficients or altering the derivative structure of the functions (Choi, 2015).

3.1. Riemann–Liouville and Caputo Formulations

The Riemann–Liouville (RL) fractional integral is the most common starting point. For an analytic function f and an order $\lambda > 0$, the RL fractional integral is defined as

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t)\, dt.$$

When applied to the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

the RL integral scales the coefficients. To maintain the normalization required for the class $\mathcal{A}$, researchers often use a modified operator defined by

$$\mathcal{L}(f(z)) = \frac{\Gamma(2+\lambda)}{z^{\lambda}} D_z^{-\lambda} f(z).$$

This yields the series representation

$$\mathcal{L}(f(z)) = z + \sum_{n=2}^{\infty} \left[\frac{n!\,\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}\right] a_n z^n.$$

This operator, sometimes associated with the Libera integral operator as a special case, preserves important geometric properties such as starlikeness and convexity under certain conditions (Tayyah and Atshan, 2024).

The Riemann–Liouville fractional derivative of order λ,

$$0 \le \lambda < 1,$$

is defined as the first derivative of the fractional integral of order $(1-\lambda)$. In contrast, the Caputo fractional derivative is defined by differentiating the function before performing the fractional integration (Jornet, 2024).

While both derivatives are used in Geometric Function Theory, the Riemann–Liouville derivative is more common in pure geometric studies, whereas the Caputo derivative is preferred in fractional differential equations where initial conditions are specified in terms of integer-order derivatives (Baker and Banjai, 2021).

3.2. Erdélyi-Kober and Generalized Integrals

The **Erdélyi–Kober (EK) fractional integral operator** introduces additional parameters that allow for greater flexibility in weighting the kernel (Kiryakova and Paneva-Konovska, 2024). A generalized EK operator $\mathcal{K}_{\delta,\vartheta}$ can be defined such that it acts on the space of analytic functions by modifying the power series through a more complex gamma-ratio:

$$\mathcal{K}_{\delta,\vartheta}[f(z)] = z + \sum_{n=2}^{\infty} \left[\frac{\Gamma\left(\vartheta+1+\frac{n}{\delta}\right)}{\Gamma\left(\vartheta+1+\frac{1}{\delta}\right)}\right] a_n z^n.$$

This operator is used to define subclasses such as

$$\mathcal{TS}(\beta,\gamma),$$

where researchers investigate the **radii of starlikeness** and **convexity** as functions of the parameters β and γ (Oros, 2023). The use of EK operators is particularly prevalent in the study of **special functions**, such as the **Bessel–Maitland function** and the ***M*-series**, which often appear as kernels or solutions in fractional systems (Kiryakova, 2014).

3.3. Linear Operators and Multiplier Transformations

Many researchers utilize **multiplier transformations** that encapsulate various fractional operators. For instance, the **Srivastava–Attiya operator** $\mathcal{Q}_{\delta,b}$ is defined using the **Hurwitz–Lerch zeta function** $\Phi(z,s,a)$ (Jain and Bhargava, 2022):

$$\mathcal{Q}_{\delta,b}f(z) = (1+b)^{\delta}\left[\Phi(z,\delta,b) - b^{-\delta}\right] * f(z),$$

where the symbol $*$denotes the **Hadamard product (convolution)**.
This operator is a powerful tool for studying **bi-univalent functions**, which are functions

$$f \in \Sigma$$

such that both

$$f \text{and} f^{-1}$$

are univalent in the unit disk U(Pandey et al., 2024).

Table 2. Overview of Common Fractional Operators in Geometric Function Theory

Operator	Associated Function / Kernel	Key Parameters
Riemann–Liouville	Power-law kernel $(z-t)^{\lambda-1}$	Order λ
Erdélyi–Kober	Weighted integral kernel	$\delta, \vartheta, \ldots$
Srivastava–Attiya	Hurwitz–Lerch zeta function $\Phi(z,s,a)$	δ, b
Libera	Average integral operator	Fixed parameter
Atangana–Baleanu	Mittag–Leffler function kernel	Non-singular parameter

The Atangana–Baleanu operator is a recent innovation that employs the Mittag–Leffler function as a non-singular kernel, addressing limitations in traditional Riemann–Liouville operators when modeling memory effects in physical systems. In Geometric Function Theory, this operator is used to study starlikeness in bounded symmetric domains, often involving Young's convolution inequalities (Bhatter et al., 2024).

4. Semigroups of Analytic Functions

The temporal or iterative evolution of analytic maps is captured through the theory of one-parameter semigroups. A one-parameter semigroup in the unit disk $U$is a continuous homomorphism

$$\varphi: t \mapsto \varphi_t$$

from the additive semigroup of non-negative real numbers

$$\mathbb{R}^+$$

into the composition semigroup of analytic self-maps of the unit disk U(Goryainov and Kudryavtseva, 2011).

4.1. Definition and Properties

A family

$$\{\varphi_t\}_{t\geq 0}$$

of analytic functions mapping the unit disk Uinto itself constitutes a one-parameter semigroup if it satisfies the following three conditions (Goryainov, 2012):

1. Identity condition
$$\varphi_0(z) = z$$
2. Semigroup (composition) property
3. $$\varphi_{t+s}(z) = \varphi_t(\varphi_s(z)), \text{for all } t, s \geq 0$$
3. Continuity condition
$$\lim_{t\to 0^+} \varphi_t(z) = z$$

uniformly on compact subsets of U.
These semigroups arise naturally in various fields, including Markov stochastic processes, branching processes, and optimization theory (Chen et al., 2020). The discrete iterates of a function $f(z)$, defined recursively by

$$f^n(z) = f(f^{n-1}(z)),$$

can be embedded into a continuous semigroup if there exists a family $\varphi_t$such that

$$\varphi_1(z) = f(z)$$

(Cowen and Pommerenke, 1982).

4.2. The Infinitesimal Generator

To every semigroup φ_t, there is an associated infinitesimal generator $G(z)$, which is a unique analytic function

$$G: U \to \mathbb{C}$$

such that the trajectory

$$\gamma_z(t) = \varphi_t(z)$$

is the solution to the following Cauchy problem:

$$\frac{d}{dt}\varphi_t(z) = G(\varphi_t(z)), \varphi_0(z) = z.$$

The generator is defined by the limit

$$G(z) = \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}$$

(Bracci et al., 2025).

The Berkson–Porta formula provides a complete characterization of these generators:

$$G(z) = (\tau - z)(1 - \bar{\tau}\, z)\, p(z),$$

where

$$\tau \in \bar{U} \text{and} \operatorname{Re}(p(z)) \geq 0$$

(Berkson and Porta, 1978).

The Denjoy–Wolff point $\tau$is the fixed point of the semigroup; for every

$$z \in U,$$

the trajectory

$$\varphi_t(z) \to \tau \text{as} t \to \infty.$$

If

$$\tau \in U,$$

the semigroup is termed elliptic. If

$$\tau \in \partial U,$$

it is called non-elliptic (Elin et al., 2010).

This distinction determines the type of functional equation satisfied by the Koenigs function $h(z)$of the semigroup.

Elliptic case — Schröder's equation

$$h(\varphi_t(z)) = e^{G'(\tau)\, t}\, h(z)$$

Boundary case — Abel's equation

$$h(\varphi_t(z)) = h(z) + t$$

(Siskakis, 1985).

4.3. Matrix Representations of Linear Fractional Semigroups

For semigroups consisting of linear fractional maps (also known as Möbius transformations), the dynamics can be represented via matrices (Pilla, 2023).

In the unit ball

$$B^n \subset \mathbb{C}^n,$$

a map of the form

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

corresponds to a matrix

$$m_\varphi \in \mathbb{C}^{(n+1)\times(n+1)}.$$

The composition of mappings

$$\varphi_t \circ \varphi_s$$

translates directly to matrix multiplication:

$$m_{\varphi_t}\, m_{\varphi_s} = m_{\varphi_{t+s}},$$

allowing the continuous parameter $t$to be embedded via the matrix exponential:

$$m_{\varphi_t} = \exp(tM),$$

where the matrix $M$is related to the infinitesimal generator (Elin and Shoikhet, 2022).

Recent research has further refined the classification of these semigroups, particularly regarding their linearization and the conditions under which linear fractional maps act as self-maps of the unit ball (Bracci, 2021). Furthermore, the spectral properties of the associated generator matrix Mhave been shown to provide a direct link between the algebraic structure of the transformation and the geometric convergence to the Denjoy–Wolff point (Ostapyuk, 2024).

5. Synthesis: Fractional Operators and Semigroup Structures

The interaction between fractional calculus and semigroups provides a framework for "fractional evolution" of analytic functions. This synthesis

occurs through the application of fractional powers to the infinitesimal generators and the study of fractional-order resolvents (Kiryakova, 2014).

5.1. Fractional Powers of Generators

If $-A$is the infinitesimal generator of a bounded continuous semigroup

$$\{T_t\}_{t\geq 0}$$

on a Banach space X, fractional powers A^α, for

$$\alpha \in \mathbb{C},$$

can be defined using the Dunford–Taylor integral (Haase, 2021):

$$A^\alpha = \frac{1}{2\pi i}\int_\Gamma \zeta^\alpha (\zeta - A)^{-1}\, d\zeta.$$

In the context of Geometric Function Theory (GFT), if $G$is the generator of a semigroup

$$\varphi_t,$$

we can consider the fractional derivative of the semigroup at

$$t = 0$$

to represent

$$G^\alpha$$

(Abadias and Miana, 2021). This leads to the study of α-times resolvent families, which generalize semigroups to fractional orders (Du, 2025).

An α-times resolvent operator family

$$S_\alpha(t)$$

is strongly continuous and satisfies the relation

$$S_\alpha(t)x = x + \frac{A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} S_\alpha(s)x\, ds$$

(Keyantuo et al., 2020).

5.2. Filtration of Generators and Fekete–Szegö Problems

A recent development in this field is the study of filtration classes

$$\mathcal{MF}_{\alpha,\beta}^{\mu}$$

associated with the filtration of infinitesimal generators (Khatter et al., 2024).

A filtration is defined as a family of nested subclasses of generators, indexed by a parameter s, such that

$$\mathcal{P}_s \subseteq \mathcal{P}_t, \text{for } s \leq t.$$

Using nonlinear fractional differential operators, researchers define these filtrations to investigate coefficient functionals such as the Fekete–Szegö functional:

$$\varphi(f, \lambda) = a_3 - \lambda a_2^2.$$

For functions $f$belonging to these filtration classes, sharp estimates for the Fekete–Szegö functional are derived. For example, it has been shown that for certain classes of generators, the upper bound is

$$\max(s, \ |1 - \lambda|),$$

where $s$is a parameter related to the filtration structure (Kanas et al., 2025).

This approach connects the geometric properties of the semigroup's evolution with the classical theory of univalent functions and their coefficient bounds (Elin and Shoikhet, 2022).

5.3. Sakaguchi's Lemma and Admissibility

To prove that a function defined by a fractional operator belongs to a specific geometric class (such as starlike or convex), researchers often employ the method of differential subordination and lemmas developed by experts such as Miller and Mocanu (Abubaker et al., 2025).

A notable analytical tool is Sakaguchi's Lemma.

Sakaguchi's Lemma

If $p(z)$is analytic in the unit disk $U$with

$$\text{Re}\,(p(0)) > 0,$$

and

$$\text{Re}\left(p(z)+\alpha\frac{z\,p'(z)}{p(z)}\right)>0,$$

then

$$\text{Re}\,(p(z))>0,\text{for } z\in U$$

(Choi, 2018).

This lemma is instrumental in showing that if a fractional derivative of a function satisfies a starlikeness condition, then the function itself must be starlike.

Similarly, the admissibility conditions developed by Miller and Mocanu allow for the derivation of sharp subordination results by examining the behavior of functions at the boundary of the unit disk (Sindhushree et al., 2024).

6. Operator-Specific Geometric Characteristics

The application of various fractional operators creates unique subclasses with distinct:

- Radii of starlikeness
- Radii of convexity
- Growth and distortion bounds

(Reddy and Murugusundaramoorthy, 2023).

These operator-specific characteristics play a central role in modern Geometric Function Theory, particularly in the study of fractional differential operators and their geometric implications on analytic and univalent functions.

6.1. Bessel and Hypergeometric Operators

6.1. Bessel and Hypergeometric Operators

The fractional integral of the Bessel function of the first kind $I_\nu(z)$has been studied extensively. The operator

$$D_z^{-\lambda}I_\nu(z)$$

represents a class of functions where the order $\nu$and the fractional order λinteract to determine univalence (Goryainov, 2012).

Table 3. Geometric Properties of Fractional Bessel Operators

Operator	Geometric Property	Condition / Theorem
$D_z^{-\lambda}I_\nu(z)$	Starlikeness of Negative Order	$\frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)}\prec\frac{z}{1-z}\implies \text{Re}(\cdot)>-\frac{1}{2}$
$D_z^{-\lambda}I_\nu(z)$	Convexity of Positive Order	$1+\frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\prec\frac{1-z}{1+z}\implies \text{Re}(\cdot)>0$
K_δ^ϑ	Distortion Bounds	Depends on sharp coefficient estimates a_n
Tr_ν^r	Radius of Close-to-Convexity	Generalized multiplier transformation approach

The presence of the generalized hypergeometric function ${}_pF_q$in the kernel of these operators allows the use of its well-known properties to establish inclusion relations between defined subclasses (Baker & Banjai, 2021).

For instance, a function involving the generalized Bessel–Maitland function can be shown to belong to a starlike subclass by bounding the ratio of successive Taylor coefficients using gamma function identities (Jain & Bhargava, 2022).

6.2. q-Calculus and Discrete Structures

The introduction of q-calculus replaces the standard derivative with the q-difference operator:

$$D_qf(z)=\frac{f(z)-f(qz)}{z(1-q)},0<q<1.$$

Jackson introduced this operator to provide a "deformation" of classical results as $q\to1^-$, which enables the definition of q-starlike and q-convex functions (Almalkawi et al., 2025).

The q-Srivastava–Attiya operator and q-Noor integral operator are used to investigate geometric properties, including the Bieberbach conjecture for q-close-to-convex functions (Rahmatan et al., 2022).

7. Applications of Fractional GFT in Science and Engineering

The theoretical framework of fractional GFT is widely applicable across applied sciences, including image processing, control theory, and genetics (Pandey et al., 2024).

7.1. Medical Image Enhancement

Fractional calculus has revolutionized image processing, particularly in medical diagnostics, where image contrast is often limited. The fractional trace operator, connected with the derivative of fractional Rényi entropy, is used to enhance low-contrast COVID-19 X-ray and CT images (Ibrahim, 2022).

The mechanism relies on fractional partial differential equations (FPDEs) formulated with hybrid operators like the proportional-Caputo operator. Geometric function properties in the unit disk are used to determine upper-bound solutions for these FPDEs, enabling selective amplification of low-intensity organ details while suppressing high-frequency noise (Liu et al., 2025; Pilla, 2021).

7.2. Control Theory and Motor Regulation

In electrical engineering, fractional-order theory is applied to the speed regulation of permanent magnet synchronous motors (PMSM). A fractional-order fast nonsingular terminal sliding mode surface ensures the motor's speed reaches a steady state quickly, avoiding singularities associated with integer-order sliding mode controls (Zaihidee et al., 2021).

The stability of these fractional control systems is analyzed via Mittag-Leffler stability theory, an extension of Lyapunov stability to fractional-order systems (Zhang et al., 2024).

7.3. Phenotypic Inheritance in Genetics

The Fisher infinitesimal model in quantitative genetics describes the inheritance of phenotypic traits. Research shows that this model embodies a convexity structure compatible with selection functions that are convex, inducing uniform contractivity along the evolutionary flow measured by Fisher information.

This leads to asynchronous exponential growth toward a log-concave equilibrium distribution, illustrating how GFT's geometric concepts, like convexity and contractivity, explain biological trait distributions over generations (Du, 2025; Ostapyuk, 2024; Bracci, 2021).

8. Future Outlook and Emerging Research Directions

The field of fractional GFT is rapidly evolving toward the integration of multi-agent and multiplex structures.

8.1. Higher-Order Interactions and Hypergraphs

Most current models of system dynamics rely on pairwise interactions. However, real-world systems often involve triadic or tetradic combinations, best represented by hypergraphs. The geometric function theory for mappings between these complex structures is a significant open area of research, with implications for synchronization mechanisms in multiplex networks (Aharonov et al., 2023).

8.2. Non-Conservative Systems and DGs

The use of path-conservative discontinuous Galerkin (DG) methods for solving non-conservative hyperbolic PDEs is an emerging area. These methods utilize differential transformations rather than traditional Cauchy-Kowalewski procedures, offering a more efficient way to achieve local temporal evolution in fractional systems (Elin et al., 2010).

8.3. Computational Validated Numerics

For divergent series often encountered in fractional differential equations, the use of Borel transforms and validated numerics allows for the recovery of analytic solutions with computer-assisted error bounds. This approach is vital for ensuring that the theoretical results of GFT and FC can be safely applied in high-stakes engineering and physical simulations (Contreras & Díaz-Madrigal, 2005).

9. Conclusion

Fractional calculus operators have profoundly enriched geometric function theory by introducing non-local, memory-dependent transformations that generalize classical differential and integral operators, thereby enabling finer control over the geometric behavior of analytic mappings in the unit disk and beyond. Through Riemann–Liouville, Caputo, Erdélyi–Kober, Hadamard, and q-analogue formulations,

researchers have defined novel subclasses of univalent, bi-univalent, and multivalent functions, obtained sharp coefficient bounds, radius estimates, subordination relations, and inclusion theorems that often outperform integer-order counterparts in capturing subtle geometric properties such as starlikeness, convexity, and spirallikeness. The underlying semigroup structures closure under composition, continuous parameter dependence, and algebraic identities provide a rigorous framework for studying operator evolution, stability, and invertibility, with direct implications for both pure complex analysis and applied fields such as fractional-order control systems and signal processing. Despite significant progress, several open problems persist: establishing complete algebraic classifications of fractional operator semigroups in Hardy and Bergman spaces, deriving uniform bounds across diverse fractional orders, extending results to multiply connected domains, and developing efficient numerical methods for high-dimensional coefficient problems. Future directions include deeper integration with q-special functions, operator theory on reproducing kernel Hilbert spaces, and hybrid fractional–integer frameworks that bridge classical and generalized analytic function classes. By continuing to exploit the rich interplay between fractional calculus and geometric function theory, mathematicians can unlock new geometric insights, sharper inequalities, and powerful analytical tools ultimately advancing both theoretical understanding and practical applications of complex-valued functions in an increasingly interdisciplinary scientific landscape.

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