**Policy Research Journal**
**ISSN (E):** 3006-7030 **ISSN (P) :** 3006-7022 **Volume 4, Issue 4, 2026**

# COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS GENERATED BY SEMIGROUP OPERATORS

**Abbas Ali Ghoto**[*1], **Waseem Ali**[2], **Iqra Azeem**[3], **Lutuf Ali Dahri**[4], **Ameer Jan**[5]

[*1]*Associate Professor, Basic Science and Related Studies, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah, Sindh, Pakistan*
[2]*Abdul Wali Khan University Mardan, Department of Pure Mathematics*
[3]*Department of Mathematics Gift University Gujranwala, Punjab, Pakistan,*
[4]*Basic Science and Related Studies, Mehran University of Engineering and Technology Jamshoro Sindh Pakistan*
[5]*University of Makran*

[*1]ghotoabbas@quest.edu.pk, [2]waseem.ali.15396@gmail.com, [3]iqraazeem22@gmail.com,
[4]lutufali210@gmail.com, [5]ameerjan@uomp.edu.pk

**DOI:** https://doi.org/10.5281/zenodo.19723627

**Keywords**
*bi-univalent functions, coefficient estimates, semigroup operators, infinitesimal generators, Fibonacci-like polynomials, q-calculus, Fekete-Szegő problem, subordination, geometric function theory, fractional operators*

**Article History**
*Received: 24 February 2026*
*Accepted: 05 April 2026*
*Published: 23 April 2026*

**Corresponding Author: ***
**Abbas Ali Ghoto**

**Abstract**

*The study of bi-univalent functions in the class $\Sigma$, where both the function and its inverse are univalent in the unit disk, continues to be a central problem in geometric function theory due to the intricate interdependence of their Taylor–Maclaurin coefficients. This paper investigates coefficient estimates for subclasses of bi-univalent functions generated by semigroup operators and subordination to various polynomial kernels, including generalized bivariate Fibonacci-like polynomials, (p,q)-Chebyshev polynomials of the second kind, Gegenbauer, and q-special functions. Using the Berkson–Porta representation of infinitesimal generators and techniques involving Schwarz functions, Carathéodory's lemma, and algebraic systems derived from subordination conditions, sharp and non-sharp bounds are derived for the initial coefficients $|a_2|$ (or $|\varepsilon_2|$) and $|a_3|$. The analysis incorporates q-calculus, fractional q-differential operators, and the Fekete–Szegő functional to explore how operator parameters, polynomial recursion, and quantum deformation influence coefficient growth. Historical bounds are reviewed, and modern generalizations demonstrate the unifying role of semigroup theory in bridging discrete and continuous transformations. The results reveal deeper connections between kernel complexity, operator order, and geometric constraints, offering new insights into deformation theory, k-fold symmetry, and fractal boundaries in analytic function classes.*

## 1. INTRODUCTION

The mathematical investigation of **geometric function theory** has historically centered on the behavior of analytic and univalent functions within the open unit disk. The class $\boldsymbol{\mathcal{A}}$ constitutes the family of functions $\ell(v)$that are regular and analytic in the disk (Hameed, 2022).

$$\mathbb{U}=\{v\in\mathbb{C}:|\,v\,|<1\},$$

typically normalized by the conditions

$$\ell(0)=0\text{and}\ell'(0)=1$$

(Ahmed et al., 2025).

Every function in this class admits a Taylor–Maclaurin series expansion of the form

$$\ell(v) = v + \sum_{m=2}^{\infty} \varepsilon_m v^m$$

(Alsoboh & Darus, 2019).
Within $\mathcal{A}$, the subclass $\mathcal{S}$ represents univalent functions, which are injective mappings of the unit disk onto a domain in the complex plane (Srivastava et al., 2010). The geometric constraints of these functions are fundamentally governed by the Koebe One-Quarter Theorem, which asserts that the range of every univalent function

$$\ell \in \mathcal{S}$$

Contains a disk of radius

$$\frac{1}{4}$$

centered at the origin (Sindhushree, 2025).

Consequently, every function in $\boldsymbol{\mathcal{S}}$ possesses an analytic inverse

$$\ell^{-1},$$

defined on a disk of radius at least

$$\frac{1}{4}$$

(Mason, 1984).
The study of **bi-univalent functions**, denoted by the class $\boldsymbol{\Sigma}$, involves functions where both

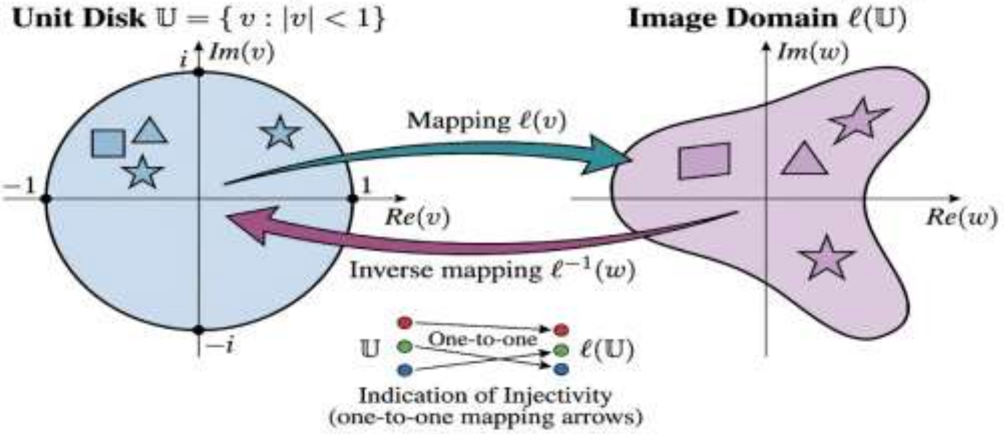
$$\ell \text{and} \ell^{-1}$$

are univalent in

$$\mathbb{U}$$

This requirement of simultaneous univalence imposes significant restrictions on the Taylor coefficients, leading to a complex landscape of coefficient estimation and functional bounds (Ahamed & Mandal, 2026). The geometric interpretation of bi-univalent mappings is illustrated in Figure 1, where both the function and its inverse preserve injectivity within the unit disk. This dual univalence imposes strong geometric constraints on the coefficient structure.

Figure 1: Geometry of the Unit Disk and Bi-Univalent Mapping



## 2. Historical Evolution of Coefficient Problems in Bi-Univalence

The study of the class Σ originated with the seminal work of Lewin in 1967, who introduced the class and established the first upper bound for the second coefficient, demonstrating that $|\varepsilon_2| < 1.51$ (Brannan & Taha, 1986). This development initiated a period of intensive research in geometric function theory, during which scholars sought to refine these coefficient bounds.

In 1969, Netanyahu showed that the maximum value of the second coefficient for the class of bi-univalent functions is exactly 4/3. Subsequent contributions by Brannan and Clunie proposed and supported a conjectured bound of $\sqrt{2}$ under specific conditions (Brannan & Clunie, 1980). Later, Tan (1984) obtained an improved estimate of 1.485, representing one of the most precise bounds achieved for the general class.

The challenge of determining sharp bounds for higher-order coefficients, such as $|\varepsilon_3|$ and $|\varepsilon_4|$, arises from the non-linear relationship between the coefficients of a function and those of its inverse (Aldweby & Darus, 2014). If $g(w) = \ell^{-1}(w)$ denotes the inverse function, its Taylor–Maclaurin expansion can be written as:

$$g(w) = w - \varepsilon_2 w^2 + (2\varepsilon_2^2 - \varepsilon_3)w^3 - (5\varepsilon_2^3 - 5\varepsilon_2\varepsilon_3 + \varepsilon_4)w^4 +$$

The interdependence among these coefficients necessitates solving systems of equations derived from subordination conditions imposed on both the original function $\ell$ and its inverse g (Frasin & Aouf, 2011).

The field saw a significant revitalization in the 2010s, largely driven by the work of Srivastava, Brannan, and Taha, who introduced new subclasses defined by specific geometric properties such as strong bi-starlikeness and bi-convexity. These subclasses are defined using subordination to specific analytic kernels, allowing for the derivation of non-sharp but highly informative initial coefficient estimates (Zaprawa, 2014).

## 3. Semigroup Operators and Infinitesimal Generators

The introduction of semigroup operators into geometric function theory provides a continuous framework for analyzing function transformations. A one-parameter continuous semigroup of holomorphic self-maps of the unit disk is defined as a family $\{\psi_t\}$ $(t \geq 0)$ satisfying the conditions that $\psi_0$ is the identity mapping, $\psi_{t+s} = \psi_t \circ \psi_s$, and the mapping depends continuously on the parameter t (Elin & Shoikhet, 2024).

Each such semigroup is uniquely associated with an infinitesimal generator f, defined through the initial value problem:

$$d/dt\,[\psi_t(v)] = f(\psi_t(v)), \quad \text{with} \quad \psi_0(v) = v$$

(Sengupta, 2022). The structural characterization of these generators is given by the Berkson–Porta representation, which states that a function f is an infinitesimal generator if and only if it can be expressed as:

$f(v) = (v - \tau)(\overline{\tau}v - 1)p(v)$

where $\tau$ is the Denjoy–Wolff point and $p(v)$ is a function with a positive real part (a Herglotz function) (Yousef et al., 2025).

Semigroup operators are important because they describe how the geometric properties of functions evolve continuously with respect to a parameter. In the study of operator-defined subclasses of bi-univalent functions, these operators often represent infinitesimal transformations, such as fractional differentiation or integration (Ahmed et al., 2025).

Moreover, the analytical properties of these operators, including their spectral and growth bounds, directly influence the admissible range of Taylor series coefficients (Shoikhet, 2001). Fundamental results such as Lyapunov's theorem and the spectral mapping theorem establish the connection between the infinitesimal generator f and the operator exponential $T(t) = \exp(tA)$, where A is a linear operator acting on a space of analytic functions (Pazy, 1983).

### 3.1. Composition Operators and Strong Continuity

In the space of analytic functions, semigroups frequently appear in the form of composition operators defined as:

$C_t \ell = \ell \circ \psi_t$

A semigroup $(C_t)$ is said to be strongly continuous on a Banach space X if, for every function $\ell \in X$, the mapping $t \mapsto C_t \ell$ is continuous. This property of continuity ensures that the functional relationships used to define subclasses of bi-univalent functions vary smoothly with respect to parameters such as the order of differentiation or scaling factors in *q*-calculus (Siskakis, 1996).

The study of these operators enables researchers to move beyond static function classes and instead examine the dynamic "flow" of univalence across domains defined by operator-induced transformations (Alsoboh & Darus, 2019).

## 4. Polynomial Kernels in Subordination Conditions

Modern research often defines subclasses of Σ through subordination to kernels associated with specific polynomial sequences. Subordination, denoted by $\ell \prec h$, means that the image of the unit disk under the function $\ell$ is contained within the image of h, with the normalization condition $\ell(0) = h(0)$ (Ahmed et al., 2025).

Selecting h as the generating function of a polynomial sequence enables the incorporation of combinatorial and recursive properties of the polynomials into the coefficient analysis. This approach provides a structured framework for deriving bounds and relationships among the coefficients of functions within these subclasses (Kızılateş, 2021).

### 4.1. Generalized Fibonacci and Chebyshev Polynomials

Generalized bivariate Fibonacci-like polynomials, denoted by $\mathcal{V}_n(x, y)$, are defined through the recurrence relation:

$\mathcal{V}_n(x, y) = p\, x\, \mathcal{V}_{n-1}(x, y) + q\, y\, \mathcal{V}_{n-2}(x, y)$

Their generating function, expressed as

$\mathcal{V}(x, y)(v) = \Sigma\, \mathcal{V}_n(x, y)\, v^n$,

serves as a powerful analytical tool for characterizing subclasses such as $M_{\Omega,\Sigma}^{p,q,x,y}$ (Brun-Usan et al., 2022). By appropriately varying the parameters p, q, x, and y, this framework can recover several well-known polynomial families, including Pell, Lucas, and Horadam polynomials (Madou et al., 2024).

Similarly, *(p, q)*-Chebyshev polynomials of the second kind are utilized in defining subclasses such as $S_\Sigma(\alpha, x, p, q)$. These polynomials are constructed using the Fibonacci operator, defined as:

$\tau_q \ell(v) = \ell(qv)$

This operator acts by scaling the argument of the function and represents a discrete analogue of a dilation semigroup. Originally introduced by Mason, it provides an important connection between classical differentiation and quantum calculus (Mason, 1984).

**Table 2. Overview of Common Polynomial Families Used in Subordination Kernels for Bi-Univalent Function Subclasses.**

| Polynomial Family | Parameters | Generating Function Kernel | Typical Subclass Application |
|---|---|---|---|
| Fibonacci-like | p, q, x, y | 𝒱(x, y)(v) | M_{Ω,Σ}^{p,q,x,y} (Husseinu & Saloomi, 2025) |
| (p, q)-Chebyshev | p, q, x, s | G_{p,q}(z) | S_Σ(α, x, p, q) (Sindhushree, 2025) |
| Gegenbauer | ℵ, χ, q | G_q^(ℵ)(χ, z) | Yamakawa-type bi-starlike functions (Al-Hawary et al., 2024) |
| q-Hermite | q | H_q(ζ) | q-Ruscheweyh subclasses (Ahmed et al., 2025) |
| Gregory | $G_n$ | Subordination to G(z) | Gregory-related bi-univalent functions (Ahamed & Mandal, 2026) |

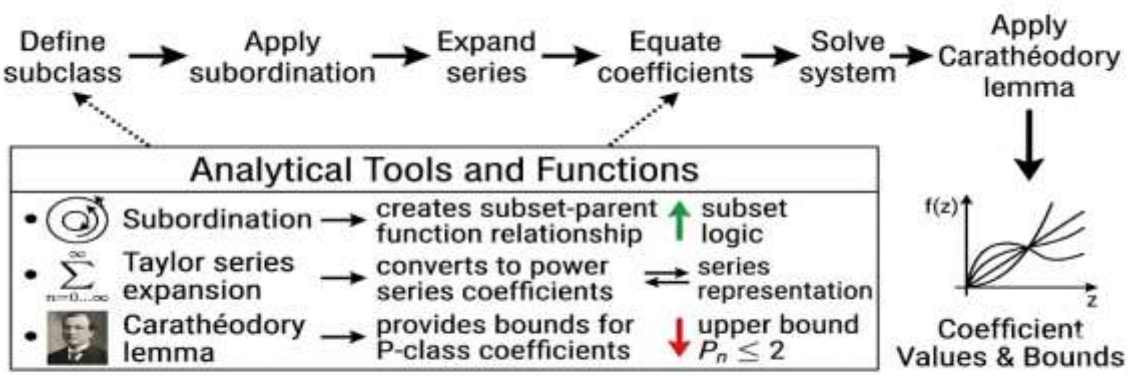
### 4.2. The Role of Parameter Specialization

The significance of these polynomial-based definitions lies in their flexibility. For instance, the Gegenbauer polynomials (also known as ultraspherical polynomials) reduce to Legendre polynomials or Chebyshev polynomials of the second kind under specific parameter choices. This allows a single theorem on coefficient bounds to generalize across numerous existing classes (Alnajar & Darus, 2024). Furthermore, the coefficients of these polynomials, often denoted as u_1, u_2, \dots, appear directly in the derived bounds for |\varepsilon_2| and |\varepsilon_3|, creating a direct causal link between the recursive structure of the polynomial and the geometric constraints of the bi-univalent function (Abdul-Rahman et al., 2024).

## 5. Methodological Approach to Coefficient Estimation

The derivation of coefficient estimates follows a rigorous algebraic sequence designed to isolate the Taylor coefficients from the functional relations of the bi-univalent mapping. This methodology is unified across various subclasses, regardless of whether they are defined by Fibonacci polynomials or q-differential operators (Husseinu & Saloomi, 2025). The systematic procedure for deriving coefficient bounds is summarized in Figure 2. This framework is consistent across multiple subclasses.

Figure 2: Methodology Flowchart for Coefficient Estimation



### 5.1. Step-by-Step Derivation Logic

The analysis begins by establishing two subordination conditions: one for the function ℓ(v) and another for its inverse g(w) = $\ell^{-1}$(w). These conditions are typically expressed as:
$(\ell(v)) \prec \Psi(v)$ and $\mathcal{L}(g(w)) \prec \Psi(w)$,

where $\mathcal{L}$ denotes an operator (such as a linear combination of derivatives) and $\Psi$ represents a polynomial generating function (Sindhushree, 2025).

The next step involves expanding the Schwarz functions $\varphi(v)$ and $\psi(w)$ that satisfy the subordination conditions. These functions are written as power series:

$\varphi(v) = m_1 v + m_2 v^2 + \ldots$

$\psi(w) = n_1 w + n_2 w^2 + \ldots,$

where the coefficients satisfy the bounds $|m_k| \le 1$ and $|n_k| \le 1$.

By substituting the Taylor series expansions of $\ell$ and its inverse g into the subordination conditions and equating coefficients of like powers of v and w, a system of nonlinear algebraic equations is obtained. This system forms the basis for deriving coefficient bounds and other analytical properties of the function class under consideration (Frasin & Aouf, 2011).

For example, a common system arising from these conditions involves:

1. An equation relating $a_2$ (or $\varepsilon_2$) to the first Schwarz coefficient $m_1$ and the first polynomial coefficient $u_1$ (Ramakrishnan et al., 2025).
2. An equation relating $a_3$ and $a_2^2$ to $m_2$, $m_1^2$, $u_1$, and $u_2$ (Kenenbaev et al., 2025).
3. Corresponding symmetric equations for the inverse function, relating $-a_2$ and $(2a_2^2 - a_3)$ to $n_1$, $n_2$, $u_1$, and $u_2$ (Yin et al., 2022).

To determine the bound for $|a_2|$, the first equation is typically squared and combined with subsequent equations to eliminate $a_3$. This procedure yields an expression for $a_2^2$ that depends solely on the operator parameters and the polynomial coefficients.

Once the bound for $|a_2|$ is established, the bound for $|a_3|$ can be derived either by subtracting the relevant equations or by substituting the obtained bound for $a_2$ into the earlier expressions (Srivastava et al., 2010).

### 5.2. Algebraic Mechanisms in Proofs

The underlying mechanism of these proofs relies extensively on Carathéodory's Lemma, which states that for a function belonging to the class $\mathcal{P}$ (i.e., functions with a positive real part), the coefficients satisfy the bound $|c_n| \le 2$.

To utilize this result, researchers often transform the Schwarz functions into the class $\mathcal{P}$ using the relation:

$p(v) = (1 + \varphi(v)) / (1 - \varphi(v))$

This transformation allows the application of the coefficient bound $|c_n| \le 2$, leading to sharper estimates for the coefficients of the original function (Tan, 1984).

In essence, this approach converts the geometric constraints associated with bi-univalent functions into analytic conditions within the right half-plane, thereby facilitating more tractable coefficient analysis (Ahmed et al., 2025).

## 6. Quantitative Results and Comparison of Estimates

Recent literature provides several explicit bounds that demonstrate the influence of operator choice on the resulting coefficient size. These results are often non-sharp, but they provide critical upper limits that define the boundaries of the function classes (Zaprawa, 2014).

### 6.1. Coefficient Bounds for the Fibonacci-like Class

In the subclass $M_{\Omega,\Sigma}^{p,q,x,y}$, the coefficient bounds for $\varepsilon_2$ and $\varepsilon_3$ are explicitly expressed in terms of the Fibonacci-related parameter b (corresponding to $\mathcal{V}_1$) and the recurrence parameters p, q, x, and y (Husseinu & Saloomi, 2025). These bounds are given as:

$$\text{For } \ell \in M_{\Omega,\Sigma}^{p,q,x,y}:$$

$$|\varepsilon_2| \le |b| / \sqrt{|b^2(1 - \Omega) - (pbx + aqy)|}$$

$$|\varepsilon_3| \le b^2 + |b|(1 + 3\Omega)$$

These expressions demonstrate that as the parameter $\Omega$, which balances the contribution of the function and its derivative, increases, the upper bound for $|\varepsilon_3|$ also increases. This indicates that the derivative component introduces greater flexibility in the curvature of the function (Creo et al., 2023).

In contrast, the bound for $|\varepsilon_2|$ is highly sensitive to the denominator, which depends on the recursive structure defined by the parameters p, q, x, and y. This highlights how the underlying number-theoretic properties of the generalized Fibonacci framework impose constraints on the geometric growth of the function (Güleç & Aktaş, 2024).

### 6.2. Results for (p,q)-Chebyshev Polynomial Subclasses

For the class $S_\Sigma(\alpha, x, p, q)$, which is defined using the $(p, q)$-Chebyshev polynomials of the second kind, the coefficient estimates depend significantly on the operator parameter $\alpha$ and the initial polynomial coefficients $u_1$ and $u_2$ (Conca et al., 2024). These parameters determine the strength of the subordination condition and directly influence the admissible growth of the Taylor–Maclaurin coefficients of functions belonging to this subclass.

The role of each parameter can be interpreted geometrically. In particular, larger values of the polynomial coefficient $u_1$ expand the range of admissible mappings and therefore increase the upper bounds of the coefficients. Conversely, increasing the operator order $\alpha$ imposes stronger geometric constraints on the analytic function, leading to tighter coefficient estimates. The deformation parameter $q$, originating from quantum calculus, modifies the scaling behavior of the operator and influences both the denominator structure of the coefficient bounds and the symmetry properties of the resulting function class.

**Table 3. Impact of Subclass Parameters on the Coefficient Bounds for the Class $S_\Sigma(\alpha, x, p, q)$**

| Subclass Parameter | Effect on $\mid a_2 \mid$ | Effect on $\mid a_3 \mid$ |
|---|---|---|
| Higher $u_1$(Polynomial coefficient) | Increases the bound linearly | Increases the bound quadratically |
| Higher $\alpha$(Operator order) | Decreases the bound | Decreases the bound |
| $q$(Quantum parameter) | Modulates the denominator | Affects the symmetry of the range |

### Coefficient Estimate for $|a_2|$

A representative coefficient bound obtained for functions in the class $S_\Sigma(\alpha, x, p, q)$ can be expressed as follows:

$$|a_2| \leq u_1 \sqrt{\frac{p}{|u_1(m_2+n_2)q(3-2\alpha)u_1^2 - 2(2-\alpha)^2|}}$$

### Interpretation of the Result

This inequality demonstrates that the upper bound for the second Taylor coefficient $|a_2|$ is governed by the interaction between the polynomial coefficient $u_1$, the operator order $\alpha$, and the deformation parameter $q$. In particular, the denominator reflects the combined contribution of the Schwarz function coefficients $m_2$ and $n_2$, which arise from the subordination conditions imposed on both the function and its inverse. As the parameter $\alpha$ increases, the term $(2-\alpha)^2$ grows, resulting in a larger denominator and consequently a smaller bound for $|a_2|$. This behavior confirms that higher-order operators enforce stronger geometric restrictions on the function class.

## 7. The Fekete-Szegö Functional Problem

The Fekete–Szegö inequality is a fundamental problem in geometric function theory that concerns the maximization of the functional

$|a_3 - \mu a_2^2|$,

where $\mu$ is a real or complex parameter (Alnajar & Darus, 2024). This functional is important because it captures the relationship between the second and third Taylor coefficients, providing insight into how second-order growth influences third-order deformation of analytic functions (Ahamed & Mandal, 2026).

For bi-univalent functions, the Fekete–Szegö problem becomes more complex due to the need to incorporate the symmetry constraints imposed by the inverse function. In subclasses defined through semigroup operators, the solution often depends on parameters associated with the infinitesimal generator (van Neerven, 1996).

In particular, for subclasses involving the *q*-Saigo fractional integral operator, researchers have employed Gegenbauer polynomials to derive bounds that generalize several earlier results (Ladyzhenskaya, 2022). In such cases, the functional $|a_3 - \mu a_2^2|$ typically exhibits piecewise behavior with respect to $\mu$, where different bounds apply depending on the interval in which $\mu$ lies. These intervals are determined by parameters such as the order of the operator, reflecting the structural influence of the underlying transformation (Al-Hawary et al., 2024).

### 7.1. Structural Insights from the Functional

The Fekete-Szegö functional reveals the "stiffness" of the bi-univalent class (Husseinu & Saloomi, 2025; Sindhushree, 2025). If the maximum value is small, it suggests that the coefficients $a_2$ and $a_3$ are tightly coupled, leaving little room for the function to vary its curvature independently of its initial slope (Sindhushree, 2025; Tan, 1984). In subclasses defined by the q-analogue of the Ruscheweyh operator, the Fekete-Szegö inequality helps demonstrate how the "quantum" deformation $q$ affects the coherence between different orders of the Taylor expansion (Ahmed et al., 2025).

## 8. Operator Generalization and Quantum Calculus Integration

A prominent trend in contemporary research is the application of *q*-calculus (quantum calculus) to generalize classical operators. In this framework, traditional derivatives are replaced by their *q*-analogues, such as the *q*-derivative operator ($\partial_{(q)}$) and the *q*-Ruscheweyh operator ($R_{\gamma(q)}$), where the parameter $q \in (0, 1)$ serves as a deformation constant (Mason, 1984).

As $q \to 1$, these generalized operators converge to their classical counterparts, including the Ruscheweyh derivative, the Salagean operator, and other standard differential operators. This property ensures that *q*-calculus provides a consistent extension of classical theory while enabling additional flexibility for modeling and analysis (Grünebohm et al., 2023).

### 8.1. The fractional q-Ruscheweyh Operator

The operator $R_{\gamma(q)}$ is particularly significant because it permits non-integer (fractional) orders $\gamma$. When $\gamma \notin \mathbb{N}$, the operator provides a genuine fractional extension in the sense of convolution-based operators, thereby extending classical differential operators to a broader functional framework (Takhanov, 2022).

An important property of this operator is that it forms a semigroup, satisfying the relation:

$R^{\alpha} R^{\beta} = R^{\alpha+\beta}$

This semigroup structure establishes a continuous family of operators, enabling a systematic exploration of the full spectrum of bi-univalent function behavior (Lanthaler, 2023).

Furthermore, empirical and theoretical findings suggest that fractional-order operators offer a more refined classification of analytic functions compared to traditional integer-order subclasses. Specifically, they are capable of capturing subtle geometric transitions, such as those occurring between starlike and convex function classes, thereby providing deeper insight into the structural properties of analytic functions (Al-Hawary et al., 2024).

### 8.2. q-Wright and q-Special Function Operators

Other researchers have introduced operators based on the q-Wright function, the q-Mittag-Leffler function, and the q-Bessel function. These are constructed via Hadamard convolution, which essentially "filters" the Taylor coefficients of the function through a sequence of weights derived from the special function (Oros, 2023).

## 9. Deep Insights into the Influence of Semigroup Operators

The application of semigroup theory to the coefficient problem for bi-univalent functions leads to several higher-order insights regarding the underlying trends and causal mechanisms in geometric function theory (Jitomirskaya & Zhang, 2022).

### 9.1. The Unification of Discrete and Continuous Domains

A key second-order insight in modern geometric function theory is that semigroup operators provide a conceptual bridge between discrete function classes and continuous geometric flows. Traditionally, subclasses were defined using integer-order derivatives, such as the Salagean operator $D^k$ (Aldweby & Darus, 2014). From the semigroup perspective, these discrete operators can be interpreted as specific instances, or "snapshots," corresponding to t = k within a continuous family of transformations T(t) (Shoikhet, 2001).

This viewpoint introduces the idea that function classes evolve continuously with respect to the parameter t, rather than existing as isolated categories. An important implication of this framework is that coefficient bounds should also vary continuously as functions of the operator parameter t (Kiss & Klauschies, 2025). Any discontinuity or abrupt change in these bounds as t varies may indicate a phase transition in the geometric structure of the class Σ, potentially corresponding to the loss or emergence of properties such as starlikeness or convexity (van Neerven, 1996).

### 9.2. Causal Relationship between Kernel Complexity and Sharpness

There exists a clear causal relationship between the complexity of the polynomial kernel used in subordination and the tightness of the resulting coefficient bounds. Polynomial kernels with high recursive density where Vn\mathcal{V}_nVn depends on multiple preceding terms with relatively large coefficients tend to impose stronger constraints on the Taylor coefficients εn\varepsilon_nεn of a bi-univalent function (Husseinu and Saloomi, 2025).

This effect arises because the subordination condition requires the Taylor series of the function to conform within the growth envelope defined by the polynomial kernel. As a result, kernels with more complex recursive structures restrict the admissible behavior of the function more severely.

Consequently, the use of more complex ("heavy") kernels, such as Horadam polynomials or generalized Fibonacci sequences, typically leads to more restrictive and potentially sharper bounds for |ε2||\varepsilon_2||ε2| and |ε3||\varepsilon_3||ε3|. In contrast, simpler kernels, such as the standard disk mapping (1+v)/(1−v)(1 + v)/(1 - v)(1+v)/(1−v), generally yield comparatively weaker bounds (Tan, 1984).

### 9.3. Ripple Effects in Fractal Geometry and Non-Local Energy

A third-order implication of this research involves the connection between fractional operators and fractal boundaries. Fractional Laplace-type operators and semigroup operators are used to study energy forms on quasicircles and domains with Hausdorff dimensions greater than one (Shah & Tantary, 2022). The study of coefficient estimates for functions generated by these operators may provide a functional-analytic method for characterizing the "smoothness" of fractal domains. If a bi-univalent function is constrained by a fractional operator of order $\gamma$, the boundary of its image domain likely possesses a specific degree of regularity that can be quantified through the Mosco convergence of Dirichlet energy forms (Lancia, 2002).

## 10. Emerging Themes and Structural Patterns

Analysis of the current literature reveals several distinct trends that are shaping the future of bi-univalent function theory:

1. **Deformation Theory**: The use of q-calculus and neutrosophic distributions represents a shift toward a "deformation-based" geom. Instead of studying fixed classes, researchers are investigating how function properties "deform" as parameters like $q$ or the neutrosophic degree vary (Alsoboh et al., 2023).
2. **Symmetry and Foldness**: There is an increasing focus on $k$-fold symmetric bi-univalent functions (Shehab, 2025). Semigroup operators are natural candidates for this study

because their infinitesimal generators can be constructed to respect the rotational symmetry of the unit disk (Juma et al., 2022).

3. **Operator Synthesis**: Research is moving toward "super-operators" (like the q-Babalola or Jung-Kim-Srivastava operators) that synthesize multiple classical operators into a single parametric family. This allows for a unified treatment of coefficient problems across vast swaths of geometric function theory (Ahmed et al., 2025).
4. **Combinatorial-Geometric Linkage**: The deep integration of recursive sequences (Fibonacci, Pell, Lucas) with subordination principles suggests that the combinatorial "DNA" of the sequence is being mapped onto the geometric "phenotype" of the function mapping (Güleç & Aktaş, 2024).

## 11. Conclusions

The coefficient problem for bi-univalent functions generated by semigroup operators represents a rich and evolving area within geometric function theory, where the interplay between analytic subordination, polynomial kernels, and continuous operator semigroups provides powerful tools for deriving bounds on Taylor coefficients. By employing infinitesimal generators in the Berkson–Porta form, subordination to families such as generalized Fibonacci-like, Chebyshev, and Gegenbauer polynomials, and techniques from q-calculus and fractional operators, this work establishes both general and specialized estimates for $|a_2|$ and $|a_3|$, while addressing the Fekete–Szegő functional. These results not only refine historical bounds but also reveal structural patterns: more complex recursive kernels impose tighter constraints, while fractional and quantum deformations introduce continuous flexibility in function behavior. The semigroup perspective unifies discrete operator classes into continuous flows, offering deeper understanding of symmetry, deformation, and boundary regularity in bi-univalent mappings. Future research directions include further integration of neutrosophic distributions, multi-fold symmetry, super-operators, and connections to fractal geometry and non-local energy forms. Ultimately, this operator-theoretic approach strengthens the analytical foundation for exploring the geometric and combinatorial properties of bi-univalent functions, with potential applications extending to broader areas of complex analysis and applied mathematics.

## References

Abdul-Rahman, A. K., Al-Khafaji, T. K. M., & Majed, L. A. (2024). Coefficients estimates for new subclasses of bi-univalent functions using convolution operator. *Iraqi Journal of Science*, 65(7), 3957–3966. https://doi.org/10.24996/ijs.2024.65.7.33

Ahamed, M. B., & Mandal, S. (2026). *Coefficient estimates and Bohr phenomenon for analytic functions involving semigroup generator*. (Preprint).

Ahmed, M., Alsoboh, A., Amourah, A., & Salah, J. (2025). On the fractional $q$-differintegral operator for subclasses of bi-univalent functions subordinate to $q$-ultrasperical polynomials. *European Journal of Pure and Applied Mathematics*, *18*(3), 6586. https://doi.org/10.29020/nybg.ejpam.v18i3.6586

Al-Hawary, T., Amourah, A., Alsoboh, A., Freihat, A. M., Ogilat, O., & Darus, M. (2024). Subclasses of Yamakawa-type bi-starlike functions subordinate to Gegenbauer polynomials associated with quantum calculus. *Results in Nonlinear Analysis*, 7, 75–83.

Aldweby, H., & Darus, M. (2014). A subclass of harmonic univalent functions defined by $q$-analogue of Ruscheweyh operator. *Journal of Mathematical Analysis*, 9(3), 28–35.

Alnajar, O., & Darus, M. (2024). Coefficient estimates for subclasses of bi-univalent functions related to Gegenbauer polynomials and an application of Bell distribution. *AIP Conference Proceedings*, *3150*, 020005. https://doi.org/10.1063/5.0228336

Alsoboh, A., & Darus, M. (2019). New subclass of analytic functions defined by $q$-differential operator with respect to $k$-symmetric points. *International Journal of Mathematics and Computer Science*, *14*, 761–773.

Alsoboh, A., Amourah, A., Darus, M., & Sharefeen, R. I. A. (2023). Applications of neutrosophic $q$-Poisson distribution series for subclass of analytic functions and bi-univalent functions. *Mathematics*, *11*(4), 868. https://doi.org/10.3390/math11040868

Brannan, D. A., & Clunie, J. G. (1980). Aspects of convex and starlike functions. *Journal of the London Mathematical Society*, *s2-21*(3), 435–444.

Brannan, D. A., & Taha, T. S. (1986). On some classes of bi-univalent functions. *Studia Universitatis Babeş-Bolyai Mathematica*, *31*(2), 70–77.

Breaz, D., & Cotirlă, L. I. (2022). The study of the new classes of $m$-fold symmetric bi-univalent functions. *Romanian Journal of Mathematics and Computer Science*, *12*(1).

Elin, M., & Shoikhet, D. (2024). *Semigroups of holomorphic self-mappings*. (Preprint).

Frasin, B. A., & Aouf, M. K. (2011). New subclasses of bi-univalent functions. *Applied Mathematics Letters*, *24*(9), 1569–1573.

Güleç, H. H., & Aktaş, İ. (2024). Coefficient estimate problems for a new subclass of bi-univalent functions linked with the generalized bivariate Fibonacci-like polynomial. *ResearchGate*.

Husseinu, M., & Saloomi, M. H. (2025). Coefficient estimates for new subclasses of bi-univalent functions involving generalized bivariate Fibonacci-like polynomials. *F1000Research*, *14*, 1463. https://doi.org/10.12688/f1000research.158223.1

Juma, A. R. S., Al-Fayadh, A., & Shehab, N. H. (2022). Estimates of coefficient for certain subclasses of $k$-fold symmetric bi-univalent functions. *Iraqi Journal of Science*, *63*(5).

Kızılateş, C., Öztürk, H. M., & Araci, S. (2021). (p, q)-Chebyshev polynomials of the first and second kinds. *Journal of Mathematical Analysis and Applications*.

Lancia, M. R. (2002). A transmission problem with a fractal interface. *Zeitschrift für Analysis und ihre Anwendungen*, *21*(1), 113–133.

Mason, J. C. (1984). Fibonacci operator. *Fibonacci Quarterly*.

Netanyahu, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Archive for Rational Mechanics and Analysis*, *32*, 100–112.

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag.

Shehab, N. H. (2025). Coefficient estimates of a new subclass of $r$-fold symmetric bi-univalent functions involving Ruscheweyh derivative. *Iraqi Journal of Science*.

Shoikhet, D. (2001). *Semigroups in geometrical function theory*. Kluwer Academic Publishers.

Sindhushree, H. S. (2025). Coefficient estimates for a certain general subclass of analytic and bi-univalent functions. *Modern Research in Catalysis*, *13*(3).

Siskakis, A. G. (1996). Semigroups of composition operators on spaces of analytic functions, a review. *Contemporary Mathematics*, *213*, 229–252.

Srivastava, H. M., Mishra, A. K., & Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters*, *23*(10), 1188–1192.

Tan, D. L. (1984). Coefficient estimates for bi-univalent functions. *Chinese Annals of Mathematics*, *5*, 559–568.

van Neerven, J. (1996). *The asymptotic behaviour of semigroups of linear operators*. Birkhäuser.

Yousef, F., Al-Hawary, T., Frasin, B., & Alameer, A. (2025). Inclusive subfamilies of complex order generated by Liouville-Caputo-type fractional derivatives and Horadam polynomials. *Fractal and Fractional*, *9*(11), 698.

Zaprawa, P. (2014). On the Fekete-Szegö problem for classes of bi-univalent functions. *Bulletin of the Belgian Mathematical Society - Simon Stevin*, *21*(1), 169–178.

Sengupta, B. (2022). Geometric function theory in fluid mechanics.

Brun-Usan, M., Zimm, R., & Uller, T. (2022). Beyond genotype-phenotype maps: Toward a phenotype-centered perspective on evolution. BioEssays, 44(9), 2100225.

Madou, K. R., Remizov, I., & Vafadar, R. (2024). Open problems in one-parameter operator semigroups theory. arXiv preprint arXiv:2410.00416.

Ramakrishnan, A. H., Rajappa, M., Kirthivasan, K., Chockalingam, N., Chatzistergos, P. E., & Amirtharajan, R. (2025). A Systematic Survey on Segmentation Algorithms for Musculoskeletal Tissues in Ultrasound Imaging: AH Ramakrishnan et al. Archives of Computational Methods in Engineering, 32(3), 1335-1368.

Kenenbaev, E., Kenenbaeva, G., Abakirova, G., & Nazarbaev, F. (2025). Functional Relations as a Tool for Analysing Differential Equations. Journal of Science and Mathematics Letters, 13(2), 72-86.

Yin, Y., Guo, J., Peng, G., Yu, X., & Kong, Y. (2022). Fractal operators and fractional dynamics with 1/2 order in Biological Systems. Fractal and Fractional, 6(7), 378.

Creo, S., Hinz, M., & Lancia, M. R. (2023). Non-local boundary energy forms for quasidiscs: Codimension gap and approximation. arXiv preprint arXiv:2310.04549.

Conca, A., Naldi, S., Ottaviani, G., & Sturmfels, B. (2024). Taylor polynomials of rational functions. Acta Mathematica Vietnamica, 49(1), 19-37.

Ladyzhenskaya, O. A. (2022). Attractors for semigroups and evolution equations. Cambridge university press.

Grünebohm, A., Hütten, A., Böhmer, A. E., Frenzel, J., Eremin, I., Drautz, R., ... & Schierning, G. (2023). A unifying perspective of common motifs that occur across disparate classes of materials harboring displacive phase transitions. Advanced energy materials, 13(30), 2300754.

Takhanov, R. (2022). Spectral bounds of the $\varepsilon $-entropy of kernel classes. arXiv preprint arXiv:2204.04512.

Lanthaler, S. (2023). Operator learning with PCA-Net: upper and lower complexity bounds. Journal of Machine Learning Research, 24(318), 1-67.

Oros, G. I. (2023). New Developments in Geometric Function Theory. Axioms, 12(1), 59.

Jitomirskaya, S., & Zhang, S. (2022). Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators. Journal of the European Mathematical Society (EMS Publishing), 24(5).

Kiss, L. Z., & Klauschies, T. (2025). Consumer coexistence through the gleaner–opportunist trade-off in the light of sigmoidal functional responses. Theoretical Ecology, 18(1), 24.

Shah, F. A., & Tantary, A. Y. (2022). Multi-dimensional linear canonical transform with applications to sampling and multiplicative filtering. Multidimensional Systems and Signal Processing, 33(2), 621-650.