

THEORETICAL FOUNDATIONS OF THE CONCRETE-PICTORIAL-ABSTRACT (CPA) APPROACH IN EARLY MATHEMATICS EDUCATION: INTEGRATING PIAGETIAN, VYGOTSKIAN, BRUNERIAN, AND MONTESSORI PERSPECTIVES

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Abstract

Pioneering CPA in support of young children's mathematical understanding comes out of the need to create opportunities for children to connect their learning to real-life experiences and to develop their cognition in the process. CPA has been widely adopted in the field and talked about in the literature in the different contexts as a teaching framework that guides young learners moving from real-life learning to pictorial representations in the end and to abstract forms of teaching and learning. In this paper, I have analyzed the different theories of the founders of the most popular teaching approaches and learning philosophies from the perspective of CPA. I have analyzed CPA from the perspective of four most popular teaching and learning philosophies. These are Piaget's constructive learning, Vygotsky's sociocultural theory of learning, Bruner's learning through representation, and Montessori's leaning through experiences, and I will elaborate on the main points of these theories and show how CPA is used to strengthen and develop mathematical thinking in young children. This paper will demonstrate how CPA in early childhood mathematics education is based on established theory and further develop the use of mathematical thinking in young children and strengthen their problem-solving skills.

INTRODUCTION

Developing children's numerical literacy and logical reasoning and problem-solving skills depends on quality early education in

mathematics. Children's early mathematical education has a lasting impact on their later mathematical success and their feelings towards mathematics. Young children often find

mathematics challenging if it is taught using abstract symbols and procedures. Young children need to have meaningful experiences to build their understanding. Early mathematics education research shows that children learn mathematics better when they are taught using an active teaching style with multiple representations where a connection is made between concrete and abstract ideas (Clements & Sarama, 2021). These research findings have led to the increased adoption of teaching styles that provide children with a gradual shift from concrete ideas to symbols in mathematics education.

Education in mathematics at its most basic involves the Concrete - Pictorial - Abstract (CPA) approach. The first stage of the CPA approach is the concrete stage of learning where children work with real objects and manipulatives. The second stage is where children learn to draw representations of the mathematics they are working on, and the final stage of learning is where children learn to work with symbols to represent mathematics. Children are able to develop a deep understanding of the concepts and not just rely on the rote memorization of rules. A systematic review that Azzumar and Juandi (2023) completed, showed that research studies published in the period of 2018 to 2022, documented the advantages that the CPA approach brought to children for their mathematical success and their logical reasoning and problem-solving skills.

The sophistication of the CPA method aligns with the learning constructivism model, where learning occurs when children create knowledge from their own experiences and their interactions with their environment. The development of a child's learning processes and the role of experiential learning in the teaching of mathematics can be traced back to Piaget and his theory of cognitive development. Piaget (1952) said that children assimilate knowledge through experiential learning and that children learn to think in a logical, operational way, over time and with the help of their experiences. More recent theories of cognitive development, in the context of teaching, continue to assert that young learners benefit from teaching approaches that

are aligned with their developmental levels and that create learning opportunities for children (Berk, 2018). In the CPA approach to teaching mathematics, the use of learning aids and resources allows children to learn mathematical concepts and relational expressions, before the introduction of abstract theories.

The concept of the CPA approach to teaching mathematics also draws significantly from Vygotsky's theory of cognitive development, which places a learning aid within the context of Social Learning and the importance of instructional support. Vygotsky (1978) elaborated further to say that children's cognitive development is enhanced through linguistic social interaction and the provision of instructional support by a more advanced peer. The CPA approach places learning and teaching scaffolds within the context of the Zone of Proximal Development. In a mathematics classroom that uses the CPA approach, the teacher uses questions, demonstrations, and discussions that guide students to progress from the use of concrete materials to the abstract reasoning done independently. The latest studies in the field of mathematics education continue to support the significance of instructional support and the guidance of a teacher in the development of a student's mathematical cognition (Radford, 2021).

Another theoretical framework that relates to the CPA structure is Bruner's theory of representation. According to Bruner (1966), there are three modes of learning: enactive, iconic, and symbolic. The enactive mode involves learning by doing, the iconic mode involves learning through images and diagrams, and the symbolic mode involves abstract thinking and the use of symbols. These modes are akin to the concrete, pictorial, and abstract stages of CPA. Mathematical research suggests that the ability to switch between different modes of representation facilitates an understanding of the underlying concepts of mathematics and enhances the ability to use mathematics in different situations (Carbonneau, Marley, & Selig, 2018).

The Montessori model of education also helps to establish the theory of CPA. It involves learning

by doing, self-discovery, and the use of materials. Montessori (1967) shows that learning materials can help children develop their thinking through their senses. This closely relates to the concrete stage of CPA, where children learn mathematical concepts through the manipulation of objects. Newer research in early childhood education suggests that children's cognitive development and ability to engage with mathematics are enhanced through active learning and physical movement within the learning environment (Zosh et al., 2022).

Recent research indicates that math education benefits from CPA-based instruction. With CPA, learners demonstrate improved math representation, stronger conceptual understanding, higher self-confidence, improved problem-solving skills, and more. Putri et al. (2020) stated CPA-based instruction helped students with math by conceptualizing the connections between physical, visual, and abstract math. Likewise, Yuliyanto et al. (2019) stated that the interaction of students' self-efficacy with CPA instruction led to better students' perceptions of math learning. Thus, CPA is a teaching model and a theory with a framework that embraces the development of learners' cognition, emotions, and concepts.

Background of the Study

Foundations in mathematics during early education assist in the cognitive growth and problem-solving abilities of children, as well as their overall future academic success. In the early years of schooling, children are taught the concepts of mathematical learning such as logical thinking, and number sense, as well as the concepts of patterns, spatial learning, and measurements. An issue that many children face in early schooling is the introduction of abstract symbols to represent mathematics and the procedural learning that often follows. Young learners often encounter great challenges as many of these symbols and procedures are introduced without the chances for exploration and without meaningful experiences. The early childhood mathematics education research focuses on the learning of abstract mathematics concepts by

children through the use of manipulatives and other visual aids, and the importance of active learning (Clements & Sarama, 2021).

Mathematics education has placed a strong emphasis on the effective teaching of newer instructional approaches. Studies show that young children need mathematics instruction that helps move them gently from informal understandings of mathematics to formal reasoning. Representation-based approaches have value, as they provide the means by which children can connect their physical experiences to the ideas of mathematics and its symbolic representations. The Concrete-Pictorial-Abstract (CPA) approach is among the most researched and acknowledged of these approaches.

The CPA method stems from the premise that children should first engage with mathematical ideas using physical materials. Then, they should depict those experiences with drawings. Finally, they should utilize abstract symbols. This sequence coincides with the ideas of Piaget, Vygotsky, Bruner, and Montessori concerning the advancement of cognitive development, especially the engagement of a child in active learning promoted through instructive experiences and social assistance with structures that abstract thinking. This sequence also offers a defensible structure through active learning that supports the development of early mathematics.

Importance of Early Mathematics Education

Early mathematics education builds essential knowledge and skills for continuous mathematical learning throughout life. Studies show early mathematics skills predict children's future academic success and their skills in structuring and solving mathematical problems (Clements & Sarama, 2021). Early mathematical education aids children's confidence in reasoning and applying mathematics in real-life contexts.

In contrast to conventional education that emphasizes memorization and rote learning, good early mathematics education focuses on understanding concepts and making connections. Learners benefit from the opportunity to deal with mathematical concepts in an active, engaging manner and through a variety of

representations and discussions. Activities that help children's reasoning and problem-solving skills, such as sorting, classifying, and comparing, contribute to the development of cognitive skills beyond mathematics. Recent studies focus on the inclusion of a variety of learning experiences and the connection among the movement of the body, thinking with images, and abstract thought in the realm of mathematics, in the instruction of early mathematics in a developmentally appropriate manner (Zosh et al., 2022).

Overview of the Concrete-Pictorial-Abstract (CPA) Approach

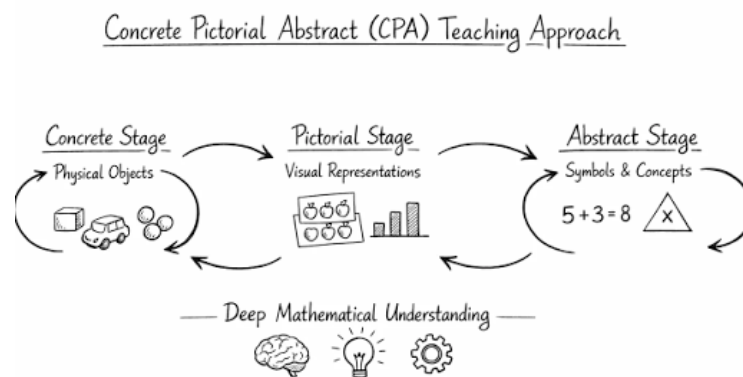


Fig 1.1 Overview of the Concrete-Pictorial-Abstract (CPA) Approach

The CPA method is centered around Bruner's idea on representation and describes learning through the three stages of enactive, iconic, and symbolic (Bruner, 1966). The concrete, pictorial, and abstract stages of CPA correspond to Bruner's three ideas of learning and are a demonstration of why CPA gives a better cognitively structured approach to learning for young students.

Recent studies, including Azzumar and Juandi (2023) and Putri et al. (2020), have shown teaching using CPA improves students' understanding of math and how to solve problems. Teaching with CPA allows students to connect ideas and concepts better, and encourages active classroom learning and visual reasoning.

The Concrete-Pictorial-Abstract (CPA) method is a phased, gradual way of teaching math that allows students to understand concepts better using different forms of expression. The first phase is the concrete stage, where students use physical objects or materials. This lets students understand math concepts through hands-on experiences. The pictorial stage lets students express experiences through drawings or visual images. The last stage introduces symbols and formal parts of math like equations.

Purpose and Scope of the Paper

This paper aims to provide an integrated theoretical analysis of Piaget, Vygotsky, Bruner, and Montessori and of the CPA method in early math education. It examines the theories of these educators and the ways in which the CPA method assists the developmental and cognitive needs of children in the area of math.

This paper also examines the relationship of the CPA method and the theories of the cognitive development of children through direct and indirect experiences, the social and cultural theories of learning, and the representation theories of education. This paper also examines the role of the teacher and child in the formative and matures stages of mathematical reasoning and the role of teacher direction and child

initiative in the development of abstract reasoning.

This paper is principally focused on theory and not on practice. It does not focus on a particular classroom practice, and it explains the theory and practice of the CPA method, based on the research and practice of the last few years. This paper also attempts to integrate older theories and contemporary theories and practices of the last few years (2018-2026) in order to explain the importance and place of the CPA method in early math education.

THE CONCRETE-PICTORIAL-ABSTRACT (CPA) APPROACH: AN OVERVIEW

The Concrete-Pictorial-Abstract (CPA) approach is a system that helps build students' understanding of math through a gradual process of incorporating physical experiences, visual aids and finally the use of abstract concepts. Young students often build their understanding of math better when they use physical objects to explore ideas before they learn to write and use mathematical concepts symbolically. Rather than using only abstract concepts and rules to teach math, CPA helps students make substantial associations between their experiences, the world, and different visual aids, along with the vocabulary of math.

The concrete stage of CPA uses physical objects and math manipulatives to help students initiate interactions with a math ideas. Activities that involve counting, sorting, or comparing help children begin to understand quantity and the relationships that exist in math. The pictorial stage builds on children's experiences and understanding through the use of drawings and/or diagrams. These visual aids help students begin to think more abstractly about ideas in math. In the final abstract stage, students are now ready to begin using mathematical symbols, numbers, and the language of math, as they have developed a strong understanding through the use of concrete and pictorial aids.

Constructivist theories attribute the development of knowledge via interaction to the learner. The constructivist perspective is seen in Bruner and Piaget's theories, where Bruner uses representations in learning as an example of the progressive stages of learning from the concrete, to pictorial, and finally to the abstract. Piaget illustrates the significance of the concrete in the development of logical thinking. The use of CPA in early childhood education is evidenced by these theories.

Recent studies continue to show the positive impact of CPA on the learning of mathematics. According to Azzumar and Juandi (2023), conceptual understanding in students is reached by connecting the disparate representations of a mathematical concept through CPA. Use of concrete representations and/or pictorial representations as reported in Putri et al. (2020) improves students' mathematical reasoning and their ability to solve mathematical problems. CPA is a theoretically grounded method and is not a teaching technique.

In early childhood education, CPA has enabled teachers to break down difficult concepts into simple ideas. The gradual progression from concrete to the abstract allows easy understanding of the concept and minimizes the cognitive gap. The CPA framework is a union of theory and practice and is a significant addition to the development of mathematical concepts for early childhood learners.

Concept and Principles of CPA

The Concrete-Pictorial-Abstract (CPA) method is a teaching framework that helps teachers facilitate the slow and progressive transition of learners from the concrete to the abstract nature of ideas. The main component of CPA is that learners will first explore ideas using real items, then from those ideas, create pictures, and finally, abstract the ideas using numbers and operations. This framework helps learners relate their experiences to the formal structure of mathematics and helps them avoid rote memorization of mathematical concepts.

Table 2.1 Hypothetical Distribution of Students' Understanding Across CPA Stages

CPA Learning Stage	Students Demonstrating Understanding	Percentage (%)
Concrete Representation	36	90%
Pictorial Representation	32	80%
Abstract Representation	29	72.5%

The CPA method employs the idea of progressive abstraction. The concrete stage uses real items, aids the development of ideas, and employs the use of manipulatives like counters, blocks, and other items. The pictorial stage allows the learners to represent the ideas and the abstract stage allows the learners to use the symbols and operations on their own. The movement from the concrete to the abstract stage follows the principles of cognitive development and constructivist learning theories.

Currently, there is a lot of research being developed, that indicates that CPA is effective, creating less cognitive load to the learners, as they are able to develop understanding progressively with each step. Azzumar and Juandi (2023) described that CPA helps the learners connect and integrate multiple representations and that

aids in the development of deep and strong conceptions of mathematics. The CPA method encourages reasoning and problem-solving, and active participation in the learning process, as opposed to a focus on procedural correctness.

The Role of Concrete Representations in Mathematics Learning

Fundamental to the CPA approach, concrete representations ground students' learning of mathematical concepts through physical opportunities. Young learners build familiarity and confidence with conceptual mathematical ideas if they first learn through represented objects. Concrete resources give learners direct experiences to navigate and make sense of relationships between quantities, patterns, and operations.

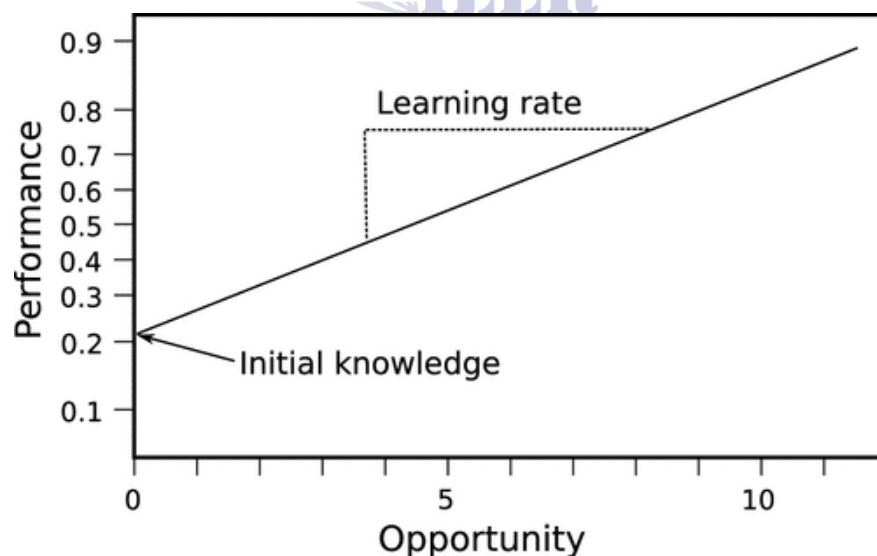


Fig 2.1 The Role of Concrete Representations in Mathematics Learning

Using resources to represent mathematical concepts provides learners with opportunities to practically reason through ideas and operations that, otherwise, are abstract and invisible. For instance, the use of resources to represent

quantity leads to the development of a range of operations through the recognition and application of addition and subtraction as quantity changes.

Concrete learning is valuable and children's learning at this stage, according to Piaget's learning theory, is through the child's interactions and active involvement with their environment (Piaget, 1952). The hands-on research of learning, whose recent findings show the use of materials on the learning of mathematics, engage learners, deepen understanding, and extend learners' abilities on the concepts to a range of application, reinforce that concrete representations are the foundation of children's mathematical thinking and its advancement (Clements & Sarama, 2021).

The Role of Pictorial Representations in Concept Development

Mathematics has both real and abstract components. Pictorial representations bridge the two. Initial experiences often require models that are concrete. Visual representations organize and express students' thoughts. Students often reflect on their thinking when they have materials that are less concrete, such as pictures, number lines, diagrams and models.

The pictorial stage helps students think about mathematics in ways that are not based on objects. Visual models help students think about mathematics in ways that are not based on objects. Students can begin to see patterns, relationships, and structures in visual models. This stage of learning encourages students to think about ideas in mathematics and form

representations of their thoughts to help them visualize the concepts.

There is a positive relationship between learning and pictorial representation. This learning stage is strongly associated with the iconic stage of Bruner's cognitive representation theory (Bruner, 1966). This stage of representation links action learning to the learning of symbols. Recent studies show that visual aids help create flexible and adaptable thinking by supporting reinterpretation of a problem from different points of view (Carbonneau, Marley, & Selig, 2018). In the CPA approach, pictorial aids precede abstract aids.

Transition from Pictorial to Abstract Mathematical Thinking

The last stage of the CPA approach describes the transition learners make of thinking abstractly by understanding and using mathematical symbols and formulas. However, this is not the same as drawing a picture and then simply writing numbers. This transition actually involves the understanding of what mathematical symbols and relationships represent.

If students are going to be successful in this transition, then strong links need to be built through their past experiences of concrete and pictorial representations to where they are now with abstract and symbolic representation. Students that understand the meaning of mathematical symbols are able to apply mathematical concepts to unfamiliar problems.

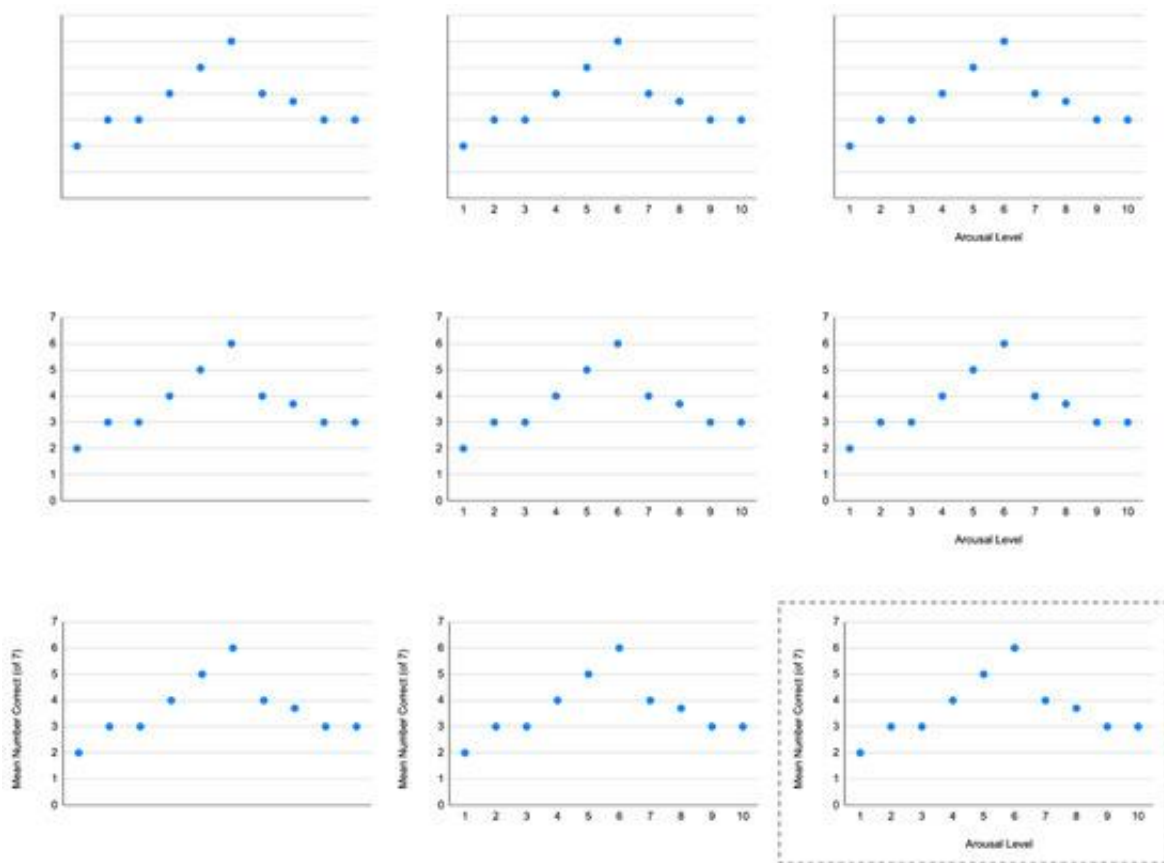


Fig 2.2 Transition from Pictorial to Abstract Mathematical Thinking

This transition of students from pictorial representations to the use of abstract symbols is described as the transition to the symbolic stage of learning by Bruner (Bruner, 1966). This transition can be described by the use of a scaffolding and a zone of proximal development framework of Vygotsky (Vygotsky, 1978) and is supported by research in mathematics education that argues an active transition to more abstract concepts improves a student's ability to understand a concept as opposed to them relying on memorization (Radford, 2021).

CPA Approach in Early Mathematics Classrooms

In early math classrooms, the CPA method gives teachers a way to introduce basic math through a developmentally appropriate framework. It helps younger students begin to understand math concepts without having to work with symbols. Teachers can implement CPA in number sense and in addition, subtraction, multiplication, division, measurement, geometry, and problem-solving.

Table 2.2 Hypothetical Comparison of Traditional and CPA-Based Mathematics Learning

Learning Approach	Students Achieving Conceptual Understanding	Percentage (%)
Traditional Instruction	24	60%
CPA-Based Instruction	36	90%

For CPA to be successfully instituted, teachers must design learning experiences that guide students through each of the levels of

representation. In order to best design these learning experiences, teachers must act as facilitators by designating the best materials,

promoting discussion, asking the right questions, and guiding students through of concrete experiences to abstract thinking.

Learning through exploration and the senses is emphasized by Montessori methods, which CPA also supports, through the use of materials designed with care. Active learning, through participation and learning that is meaningful, has shown to improve students' mathematical confidence and engagement in CPA classrooms (Putri et al., 2020; Yuliyanto et al., 2019). Early math education can benefit greatly from CPA due to the application of the theory of learning to the classroom.

THEORETICAL FOUNDATIONS OF THE CPA APPROACH

The Concrete-Pictorial-Abstract (CPA) approach is based on recognized models of child development and learning. Effectiveness of early CPAs can be examined through models of cognitive development and learning, as well as representation and experience. Additional perspectives of more recent theorists, such as Piaget, Vygotsky, Bruner, and Montessori, aid the understanding of how children develop their own understanding of mathematics through experiential learning, social interactions, the use of different forms of representation, and a gradual transition toward abstraction. All these models show that learning mathematics involves the construction of meaning through experiences that are developmentally appropriate.

Piagetian Constructivist Perspective

Some of the cornerstones of Piaget's theory of cognitive development can be utilized to explain

how children develop an understanding of mathematics. Children use their surroundings to develop knowledge. As Piaget (1952) describes, cognitive development happens when children use the processes of assimilation and accommodation. This theory is consistent with the CPA approach. Piaget considered the way children perceive the world and how children construct knowledge in separate stages. Piaget's theory of cognitive development suggests that children need to be given the right learning stimuli, the right learning opportunities and learning experiences, so that children can construct knowledge at that particular level. Learning and understanding abstract concepts of mathematics poses the need to have tangible Learning aids and the opportunity to interact and deal with them. Recent studies in early math education have stressed the need to create appropriate learning experiences that offer children the opportunities to explore and manipulate mathematical ideas (Berk, 2018).

Cognitive Development and Mathematical Learning

Piaget's theory stages cognitive development, showing that children's understanding of mathematical concepts gradually evolves at different stages. Early in the preoperational stage, children advance from primitive thinking to developing concrete operational logic. This stage of logic is evident in children's movements toward thinking of things in the world around them through real, physical objects and phenomena (Piaget, 1952).

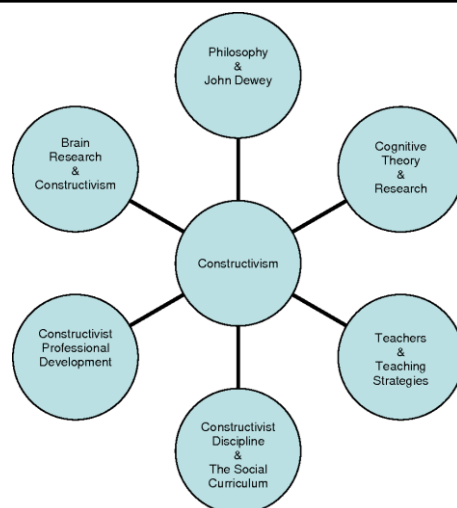


Fig 3.1 Cognitive Development and Mathematical Learning

In the development of mathematical thinking, preoperational children advance from having concrete experiences to thinking abstractly. Actually, children learn the concept of addition much better when they combine groups of physical objects before learning the equation. Following the Piagetian development of thinking, the CPA approach allows children to first physically understand the mathematical relationships, to later recognize them in a drawn form, and finally to appreciate them in their symbolic form.

Developmental alignment is a critical constituent of effective mathematics instruction. Clements and Sarama (2021) identified the importance that teaching young learners mathematics in alignment with their cognition and structuring

gradually more complex learning tasks has to the development of a strong mathematics foundation.

Concrete Experiences and Knowledge

Construction

Children learn through experiences and interactions in their environments. They learn through engagement, not passively through teaching. As children interact with and manipulate objects in their environments, they build mental schemas with the cognitions they receive. The concrete stage of CPA, which emphasizes manipulatives, aligns with this principle, as learners use objects to build and develop their foundational understandings of mathematics.

Table 3.1 Hypothetical Improvement in Mathematical Skills Through CPA

Mathematical Skill	Before CPA Implementation (%)	After CPA Implementation (%)
Number Sense	55%	85%
Problem Solving	50%	82%
Mathematical Reasoning	58%	88%
Representation Skills	52%	90%

Concrete materials help children understand concepts of quantity, comparison, patterns, and operations in a significant and contextual way. These experiences help construct the mental frameworks needed to do mathematics on an abstract level. Students without an understanding

of concrete mathematics may understand how to perform computations, yet lack a comprehensive understanding of the processes behind the computations.

Current educational research still explores the relationship between the construction of

knowledge and concrete experiences. Zosh et al. (2022) observed that active learning methods help children understand concepts at a deeper level. The connection of physical engagement with cognitive development causes this deeper level of understanding. The concrete stage of CPA is an example of this in practice. Piaget's principles of learning and constructivism are thus illustrated.

Vygotskian Sociocultural Perspective

Vygotsky's sociocultural theory is another building block in understanding CPA learning. In contrast to Piaget, who placed greater weight on the developing individual, Vygotsky highlighted the role of socially mediated activity in language and thinking, and of culturally defined tools. Vygotsky (1978) suggested children learn through dealings with other people (teachers, parents, peers) who know more than they do.

Vygotsky's theory is drawn on the CPA approach because children moving from one level of mathematical representation to another often need adult help. Teachers help learners to move from physically manipulating materials to reasoning abstractly and independently by providing appropriate explanations, demonstrations and questions. CPA is therefore a socially supported learning activity and a sequence of representations.

Social Interaction and Mathematical Understanding

Vygotsky (1978) asserted cognitive development is enhanced through social interaction because children establish comprehension from social discourse and communal activities. Mathematical comprehension improves when students articulate their thoughts and reasoning and when they engage with their educators and fellow learners for constructive critiques.

In CPA classrooms, social interaction facilitates the comprehension of both the tangible (concrete) and the abstract (conceptual) by children. Discussions about objects, and what they represent in terms of numbers, or how pictures represent relationships in mathematics, facilitate children's comprehension of mathematical reasoning and discourse.

The need for collaborative learning in mathematics education has continued to be substantiated by recent studies. Radford (2021) posited that an understanding of mathematics is realized when there is discourse, and meaning is constructed collectively, and that the learning environment is interactive. This adds weight to the fact that CPA learning is more effective in constructing the learners' understanding when meaningful interactions with the teacher take place.

Table 3.2 Hypothetical Student Engagement During CPA Activities

CPA Activity	Highly Engaged Students	Engagement Percentage (%)
Working with Manipulatives	38	95%
Creating Visual Models	34	85%
Solving Symbolic Problems	30	75%
Explaining Mathematical Thinking	32	80%

Scaffolding and the Zone of Proximal Development in CPA Learning

One of Vygotsky's most prominent theories is his idea of the Zone of Proximal Development, or the gap between what learners can do on their own versus what they can do with help. Scaffolding is the term for this sort of constructive help provided by teachers that allows

learners to accomplish tasks that are more than what they are currently capable of. (Vygotsky, 1978).

Scaffolding is derived with CPA since teachers diminish support along the way as students work from the usage of concrete resources to pictorial representations to finally the use of abstract symbols. At a certain point, learners might need

concrete resources and support from the teacher, but they might eventually learn to solve problems independently using symbols.

Scaffolding is an important part of teaching early mathematics. The ability to guide learners help develop their understanding and confidence of the mathematical concepts (Clements and Sarama, 2021).

Bruner's Theory of Representation

Jerome Bruner's representation theory provides a valid basis for conceptualizing the CPA approach. Bruner (1966) suggests that children come to know through three forms of representation: enactive, iconic, and symbolic. These modes of representation show the progression children make from direct experience, to the use of pictures, to ultimately the use of abstract symbols and ideas.

Bruner's three modes of representation correlate very well with the three stages of CPA. The enactive representation and learning through direct activity correlate with the concrete stage of CPA; the use of pictures and other forms of visual representation corresponds with the iconic stage; the use of symbols, letters, and abstract language correlates with the symbolic stage. Clearly there is a justification of the CPA approach in relation to the development of mathematical concepts.

Enactive Representation and Concrete Learning

Enactive representation is learning through action and experience. Children understand the world around them by being able to do things and move objects, and do not rely on the spoken word or printed symbols as much (Bruner, 1966). This idea relates strongly to the concrete stage of CPA, e.g. where pupils use physical objects, for example counters, blocks and mathematical equipment. As they manipulate, they build relationships and understand the different quantities and operations before even seeing the formal mathematics.

Epochs of action learning have been and will continue to be a cornerstone of learning mathematics in the classroom. Evidence shows that physical interaction with mathematics has a

positive effect on thinking and understanding mathematical concepts and will support even higher levels of abstract thinking (Clements & Sarama, 2021).

Iconic Representation and Visual Learning

In understanding an idea through different forms of imagery and drawing, we see Bruner's stage of icon representation. The pictorial stage of CPA is analogous to Bruner's representation, as it assists learners in visualizing relationships in mathematics.

Bruner suggested that number lines, drawings, and diagrams allow children to visualize their patterns and organize their thinking. They help children's cognitive tools and guide their reasoning.

The development of different mathematical representations has shown that thinking visually enhances a learner's ability to process relationships. Because of this, the pictorial stage has immense value in transitioning learners to working with abstract mathematical symbolism.

Symbolic Representation and Abstract Reasoning

The top level in Bruner's framework is called symbolic representation. In this level, learners use language, symbols, and formal procedures of mathematics to show their understanding. This level is similar to the abstract level of CPA. At this level, learners deal with equations, formulas, and mathematical notations.

According to Bruner, students should not develop the abstract understanding of symbols in a conceptual vacuum. He emphasized that abstract symbols should emerge from meaningful experiences. This is the basis of the CPA approach. In this approach, learners deal with abstract symbols only after they have dealt with the concrete and the pictorial.

This level of the CPA approach enables learners to understand the why behind the mathematical procedures and not just the how. This has an effect of promoting the development of mathematical reasoning and the understanding of mathematical concepts on a long-term basis.

Montessori Educational Perspective

The Montessori educational perspective also strengthens the CPA with its focus on thoughtfully crafted learning aids, independence, and experience-based learning. Montessori (1967) argued that children learn through their senses and that children learn through exploration of the environment. Montessori's philosophy corresponds with the concrete stage of CPA, as children learn mathematics through CPA.

Montessori education understands that the first manipulation of objects and their exploration builds the first mental concepts of children, and later children learn to manipulate and explore the world of their own concepts. This idea corresponds to the CPA, from concrete to abstraction.

Hands-On Learning and Mathematical Development

Montessori education values hands-on activities for a reason. Children get to be a part of the learning process. Montessori (1967) described how children sharpen their intellect with movement and the opportunity to solve problems and make observations.

Children are better able to understand relationships, patterns, and quantities with respect to mathematics when they have hands-on activities. Right before abstract mathematical concepts are introduced, Montessori activities guide learners to first experience math with manipulatives.

Active learning has value in early childhood education. Zosh et al. (2022) stated that children's cognitive development is best achieved when they are engaged in activities that allow them to interact with materials meaningfully.

Montessori Materials and Concrete Mathematical Experiences

Children can utilize Montessori materials to form connections to math concepts through a hands-

on, sensory approach. Montessori encourages the use of number rods, beads, and other counting manipulatives. These aids assist children in visualizing math relationships and manipulating them. This is done before the children even start to think of the relationships in an abstract way.

From the earlier described Montessori math teaching materials and the concrete stage of CPA, we can see many similarities. They both adopt the approach of incorporating physical interaction into math learning. This is done to provide children the essential experiences needed for the formation of math concepts. With the aid of materials, the children become accustomed to the concepts of math and become self-reliant.

Montessori strengthens the theoretical and practical base of CPA through the use of stimulating the ideas of focusing on the integration of concrete experiences, the empowerment of the children, the freedom of exploration, and the concrete manipulation of materials in the teaching of early math.

INTEGRATING LEARNING THEORIES WITHIN THE CPA FRAMEWORK

The Concrete-Pictorial-Abstract (CPA) approach exemplifies the integration of various prevalent learning theories and cognitive development. Piaget, Vygotsky, Bruner, and Montessori collectively explain the manner by which children, by way of different learning experiences, representation, and interaction, gradually move toward the abstraction of mathematics. Although from different perspectives, they developed theories that share common principles and allow for the explanation of the development by the child of thinking of mathematics through abstraction and when they are supported. The CPA framework provides an explanation of how children are able to think of mathematics and how they are able to think about mathematics conceptually.

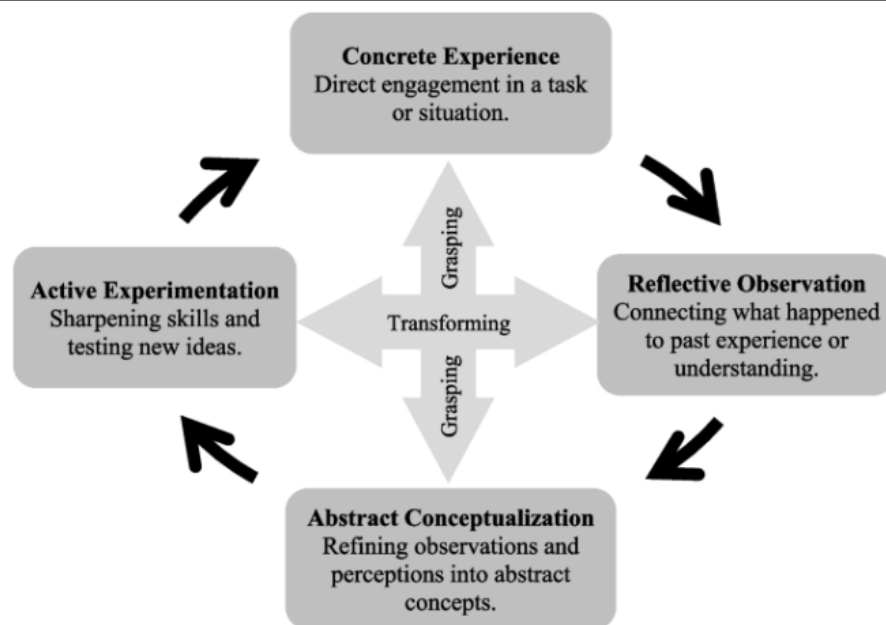


Fig 4.1 Integrating Learning Theories Within The Cpa Framework

The connection of the CPA and the above theories shows that when children are learning mathematics, they are not memorizing procedures. Children learn by way of abstract mathematics, and they learn by way of concrete objects, they learn by way of representation, and they learn by way of communication of reasoning. CPA has become an important approach in early mathematics education because of the integration of social theories, cognitive theories, and representational theories.

Constructivist Foundations of CPA

The CPA method builds on strongly constructivist theory, where learning is viewed as an active process, constructing knowledge through experience, as opposed to the passive reception of knowledge. From a constructivist stance, Piaget argues that children come to understand the world by adapting the structures of knowledge that they already have through experience and interactions with the world around them (Piaget, 1952). Montessori also aligns with this view and places a great deal of importance on the explorative and active aspects of learning in both the social and educational environment (Montessori, 1967).

Table 4.1 Hypothetical Teacher Perceptions of CPA Implementation

Teacher Response Area	Number of Teachers	Percentage (%)
CPA Improves Conceptual Understanding	45	90%
CPA Supports Student Participation	42	84%
CPA Requires Additional Planning Time	38	76%
CPA Requires Teacher Training	40	80%

The CPA approach provides the opportunity to consider learning from the perspective of the child and reflects the principles of constructivism to a great degree by encompassing the active participation of learners in mathematical tasks. In

the concrete stage, children are physically involved in the process of manipulating items and are encouraged to discover relationships and construct mathematical ideas and patterns through direct experience.

Learning that is both active and based on experience is considered to enhance young children's understanding of mathematics. According to Clements and Sarama (2021), children are better able to form and construct mathematical concepts when learning is based on active participation and well-placed learning activities that connect informal mathematical knowledge to its formal representation. The CPA approach, by design, offers the child the opportunity to construct mathematical meaning through a variety of experiences and, as such, is fundamentally a constructivist approach.

Cognitive Development and Representation in CPA

Understanding the relationship between cognitive development and representational systems is key to linking CPA and its role in learning mathematics. Gradually, abstract mathematical concepts can be understood by children, when they are offered the right form of teaching. According to Piaget, children need to have concrete experiences before they can handle reasoning on an abstract level (Piaget, 1952).

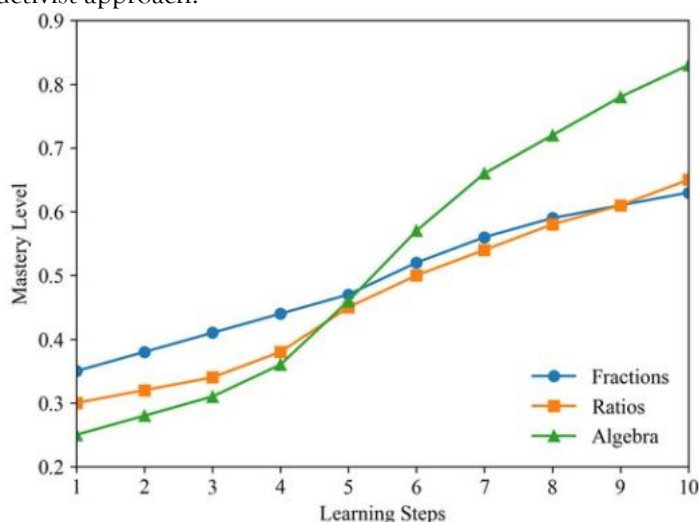


Fig 4.2 Cognitive Development and Representation in CPA

Table 4.2 Hypothetical Impact of CPA on Problem-Solving Ability

Problem-Solving Indicator	Before CPA (%)	After CPA (%)
Understanding Mathematical Problems	60%	88%
Selecting Appropriate Strategies	55%	82%
Explaining Solutions	50%	85%
Applying Knowledge to New Problems	48%	80%

Bruner's representation theory further elaborates on CPA. Learning can occur in all three of enactive, iconic, or symbolic and can be extended to the concrete, pictorial and abstract stages of CPA (Bruner, 1966). The concrete stage is a physical representation of enactive representation. The pictorial stage of CPA is indicative of learners using visual representation to assist their cognitive processing. The symbolic stage is where formal language of mathematics is introduced.

The CPA and representation theory link is useful in understanding that mathematical knowledge develops through the progressive transitions of different constructs of knowledge. Newer studies suggest that using a variety of representations assists learners in creating stronger interconnections of different mathematical concepts, and relates to the ability to transfer knowledge to new environments (Carbonneau, Marley, & Selig, 2018). CPA is grounded in providing a structured route from concrete to

abstract mathematical thinking and designed to support cognitive development.

The Role of Teachers as Facilitators of CPA Learning

Teachers are essential when it comes to successfully utilizing the CPA method because

they are the ones who guide the students through the different levels of transitional representations. With the CPA method, the teachers are facilitators that help develop learning activities, promote exploratory behavior, and help students develop reasoning in mathematics, and they are not just information givers.

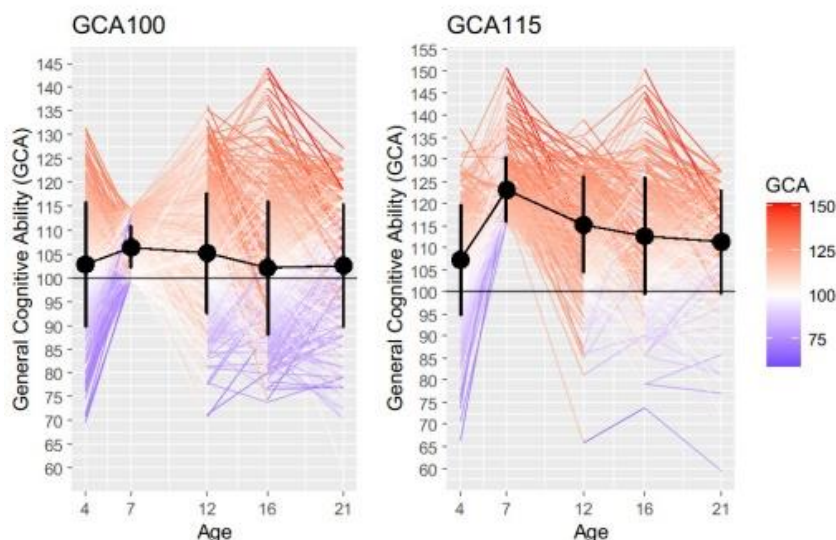


Fig 4.3 The Role of Teachers as Facilitators of CPA Learning

Table 4.3 Hypothetical Alignment Between Learning Theories and CPA Principles

Learning Theory	CPA Principle Supported	Alignment Level (%)
Piagetian Constructivism	Learning Through Concrete Experiences	95%
Vygotskian Sociocultural Theory	Scaffolding and Teacher Support	90%
Bruner's Representation Theory	Movement Between Representations	98%
Montessori Education	Hands-On Learning Materials	92%

Vygotsky's ZPD Theory places the teacher at the center of guidance and development of learners. In the words of Vygotsky (1978), learners, with the help of more knowledgeable people and with the help of scaffolding, reach a higher level of understanding. This is evident in the CPA method of teaching. When learners are helped to progress from dealing with concrete learning materials to dealing with pictorial representations, and finally to dealing with abstract Mathematical expressions, the teacher is simply providing scaffolding.

Teaching CPA requires that teachers critically design learning activities and carefully select learning materials, and consider the readiness of

the learners to progress to the next level. It also requires that the teachers recognize the differing abilities of the learners and the level of support that is required.

New studies show that teacher involvement is necessary for effective teaching of Maths because representations do not equal understanding. The development of understanding is facilitated by guided reasoning, and so students are helped by the teaching of the connections that exist among the representations (Clements & Sarama, 2021).

Supporting Conceptual Understanding Through Multiple Representations

The CPA approach builds conceptual understanding by employing multiple representations. One of the approach's most significant benefits is the representations' ability

to support understanding by making the abstract tangible. Many mathematical ideas are difficult to understand because they are difficult to see. Representations help students by making the ideas visible.

Table 4.4 Hypothetical Challenges in CPA Implementation

Implementation Challenge	Reported Frequency	Percentage (%)
Limited Teaching Materials	35	70%
Lack of Teacher Training	40	80%
Time Required for Activities	32	64%
Difficulty Managing Different Learner Levels	36	72%
Transition from Concrete to Abstract Stage	30	60%

The use of CPA's concrete, pictorial, and abstract representations helps students construct flexible thinking. Most often, procedural thinking dominates students' understanding of mathematics. Representations help students understand the relationships of various mathematical ideas. A student who has learned the addition of whole numbers and the meaning of addition with the use of concrete objects, a drawing, and an equation understands what addition means.

Bruner's representation theory and Vygotsky's theory of culturally constituted mathematical thinking support the use of different representations. Drawing from Bruner (1966), different representations support cognitive development. From Vygotsky (1978), Bruner's symbolic tools will help students structure and verbalize their thoughts.

Representation-based learning has a positive impact on students' ability to solve problems and their conceptual understanding of mathematics (Putri et al., 2020). Through the use of multiple representations, CPA helps students construct a rich understanding of mathematics.

IMPLICATIONS FOR EARLY MATHEMATICS EDUCATION

The Concrete-Pictorial-Abstract (CPA) approach is a relatively new innovation in the field of early childhood mathematics education. Its combination of cognitive theories, sociocultural

learning, representation, and experiential education is likely to encourage early childhood educators and trainers to think of ways to develop children's understanding of mathematics in a meaningful way. One of the NAS's aims is to promote understanding of mathematics as a process that goes beyond the manipulation the symbols to incorporate the whole process of thinking the problem through and making the appropriate connections between and among different representations.

The CPA approach in early mathematics encourages teachers to value participants' conceptual understanding of the mathematics as opposed to their procedural understanding. The CPA approach encourages children to move from the concrete to the pictorial and finally to the abstract in a flexible way. The principles of CPA approach of early mathematics education have a huge impact on teaching practices, curriculum design and teacher training.

Supporting Mathematical Reasoning and Problem Solving

One of the most notable advantages of CPA is that it fosters children's ability to reason and solve problems in mathematics. Children's understanding of mathematics can be compromised by conventional systems that primarily present symbols. These systems tend to be procedural in nature and explain very little about the mathematics. CPA allows children to

manipulate and develop their understanding of mathematics through structured experiences prior to formal instruction.

The concrete stage allows children to appreciate mathematical relations through the manipulation of objects and the drawing of many. The concrete stage helps children to organize their thoughts through the use of diagrams. Finally, children are enabled to manipulate symbols of mathematics and the associated operations through the abstract stage. This system develops the ability of children to reason and to interpret.

Representation-based learning helps children to solve mathematical problems because understanding relations is more important than the use of strategies which are often based on memory. This form of learning is further enhanced through the use of CPA, which promotes mathematical thinking of children by integrating numerous physical models, representations, and abstractions.

The elementary stages of learning mathematics are focused on reasoning and problem-solving, the most important skills for the learning of all other mathematics. The CPA system gives children the ability to understand mathematics and the ability to justify this understanding.

Designing Effective CPA-Based Learning Environments

To successfully apply CPA, learning environments that foster exploration, collaboration, and the progressive development of mathematical concepts must be created. The CPA classroom contains sufficient and appropriate manipulatives, visual aids, and opportunities to engage in the discourse of mathematics.

Learning activities are designed that enable children to fluidly progress through the three stages of CPA. Concrete materials should be selected, and purposeful investigations should be designed of the mathematical concept. Visual aids should be used to allow students to see patterns, relationships, and progress to the use of abstract notations.

Teachers play a critical role in designing effective CPA learning environments. For each stage,

teachers must know how, and to what extent, supports will be given as learners progress through and between the different forms. This is a demonstration of Vygotsky's scaffolding theory, where learners are supported to a level where they are able to perform the task independently (1978).

Communication and collaboration are supported elements of CPA. Students explain their ideas, compare their strategies, and develop a mathematical vocabulary as a result of mathematical discourse. Research shows that interactive classrooms and focus on the learner promotes mathematical engagement and the development of concepts (Clements & Sarama, 2021) added to the CPA classroom is the teacher's competency and classroom culture that supports rational thought and exploration.

Challenges and Considerations in CPA Implementation

The CPA approach promotes positive outcomes for early math learners. The approach does pose challenges educators must consider. One challenge is making sure educators know the CPA approach, along with the math theories connected to the approach. Without professional development, educators may use the CPA approach, along with manipulatives and/or models, but will not connect them to abstract math.

Another challenge is figuring out how quickly the concrete, pictorial, and abstract stages should take. If students spend too much time in the concrete stage, they may struggle with abstract thinking. But, if students spend too little time in the concrete stage, they may struggle with the conceptual thinking behind the procedures.

Resources in the classroom and the educational setting may limit the manipulatives, instructional materials, or professional development. Additionally, educators are challenged with varying prior knowledge of students and their learning and cultural experiences, which may require adaptation of CPA beyond the classroom context.

Newest research states, (Clements & Sarama, 2021), the different challenges and varying

developmental needs of students in the classroom have to be taken into consideration if math instruction is to be successful. Therefore, CPA must be used flexibly, based on different learner needs.

CONCLUSION

The Concrete-Pictorial-Abstract (CPA) approach is useful for the early stages of mathematics education. It delineates the link between children's experiences, representations, and abstract reasoning in mathematics. With Piaget's constructivism, Vygotsky's sociocultural theory and Bruner's theory of representation and Montessori's theory of didactics, the CPA approach is a practical reflection of the fundamental elements of cognitive and mathematical development of children.

Concrete learning develops children's initial understanding. They organize and visualize their mathematical ideas with the aid of pictorial representation. Their mathematical ideas are expressed symbolically and linguistically by abstract representation. This process aids their understanding, reasoning and problem-solving.

The theoretical integration of CPA also acknowledges the role of the teacher as a facilitator. They help students progress through the different levels of representation by providing the necessary support and scaffolding. The CPA approach scales the developmental needs of the student within a well thought out learning environment and with acting that is precise and meaningful.

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