

FINITE ELEMENT ANALYSIS OF LAMINAR MIXED CONVECTION IN A LID-DRIVEN TRAPEZOIDAL CAVITY CONTAINING FOUR INCLINED OBSTACLES

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Abstract

Mixed convection in enclosures with internal obstacles is relevant to the thermal management of electronic, energy, and process equipment, yet results for trapezoidal lid-driven cavities with several inclined obstacles are scarce. This study examines steady, laminar, two-dimensional mixed convection of an incompressible Newtonian fluid in a trapezoidal cavity (base 4L, lid 2L, height 3L) with a heated base, a cold moving lid, adiabatic inclined side walls, and four inclined rectangular obstacles. The Boussinesq-approximated equations were non-dimensionalised and solved with a penalty-formulated Galerkin finite element method in COMSOL Multiphysics on a mesh of 5,208 elements. Eight cases were examined: Grashof number $Gr = 3,000$ – $500,000$ at Reynolds number $Re = 1$, and $Re = 0.1$ – 1.0 at $Gr = 1,000$. Every case showed two counter-rotating cells on either side of a central jet. Raising Gr from 3,000 to 500,000 increased the peak dimensionless speed from 2.19 to 254.7 and drew a narrow finger of cold fluid from the lid towards the heated base; the local power-law exponent of peak speed fell from 1.47 to 0.59. At $Gr = 1,000$, raising Re from 0.1 to 1.0 reduced the peak speed from 104.37 to 1.17, produced smoother, more stratified isotherms, broke the flow symmetry, and increased the reported temperature gradient from 0.316 to 0.771. The Richardson number, $Ri = Gr/Re^2$, organised the flow morphology. Because mesh-independence and benchmark validation were not reported and Nusselt numbers were not computed, the findings are qualitative.

Introduction

Mixed convection arises when buoyancy-driven motion and externally imposed motion are of comparable importance. Its strength is commonly summarised by the Richardson number, $Ri = Gr/Re^2$, the ratio of buoyancy to inertial forcing (Bejan, 2013). The process occurs in electronic cooling, heat exchangers, solar collectors, drying, and other process equipment, where the enclosure geometry, the placement of heated and cooled surfaces, and any internal obstructions decide how heat is distributed (Bejan, 2013; Bergman et al., 2011).

The lid-driven cavity is the canonical test problem for such flows because it combines shear-driven and buoyancy-driven circulation in a simple geometry (Ghia et al., 1982). Real enclosures are seldom square and rarely empty. This study therefore considers a trapezoidal lid-driven cavity heated from below, cooled by a moving lid on top, and containing four small inclined rectangular obstacles, and uses the finite element method to describe how the Grashof and Reynolds numbers shape its velocity and temperature fields.

Research Objectives

The study pursued the following objectives:

- 1: To characterise the laminar velocity and temperature fields of mixed convection in a lid-driven trapezoidal cavity containing four inclined rectangular obstacles.
- 2: To formulate the dimensionless mathematical model and solve it with the penalty Galerkin finite element method.
- 3: To simulate the flow and heat transfer in COMSOL Multiphysics and to determine how the Grashof and Reynolds numbers alter the streamlines, isotherms, peak velocity, and temperature gradient.

Research Questions

In line with these objectives, the study asked:

- 1: How do the internal obstacles and the governing parameters, Gr and Re , shape the laminar flow and temperature fields in the cavity?
- 2: How can the governing equations be solved with a Galerkin finite element scheme?
- 3: How does the Reynolds number alter the temperature distribution and gradient between the heated and cold walls?

Literature Review

Lid-Driven Cavities and Heating Configurations

Thermal boundary conditions determine whether buoyancy opposes or reinforces the shear flow. Iwatsu et al. (1993) examined a driven cavity with a stable vertical temperature gradient, and Sharif (2007) studied shallow inclined cavities with a hot moving lid on top and a cooled bottom. Khanafer et al. (2002) treated open-ended enclosures. Basak et al. (2009) solved a square cavity with uniform and non-uniform heating of the bottom wall using a penalty finite element method, the same family of methods used here.

Modified Geometries and Internal Obstacles

Trapezoidal enclosures introduce inclined walls that alter the circulation, and Bhattacharya et al. (2013) showed that lid-driven trapezoidal enclosures can support multiple solutions. Internal bodies have been added in several forms: an elliptic obstacle in a split lid-driven trapezoidal cavity (Shah et al., 2021), a partially heated porous block (Colak et al., 2021), a porous medium with an open side (Abu-Hamdeh et al., 2020), a rotating cylinder in a wavy-walled enclosure (Alsabery et al., 2020), and a heated wavy wall in a double lid-driven cavity (Azizul et al., 2021).

Research Gap

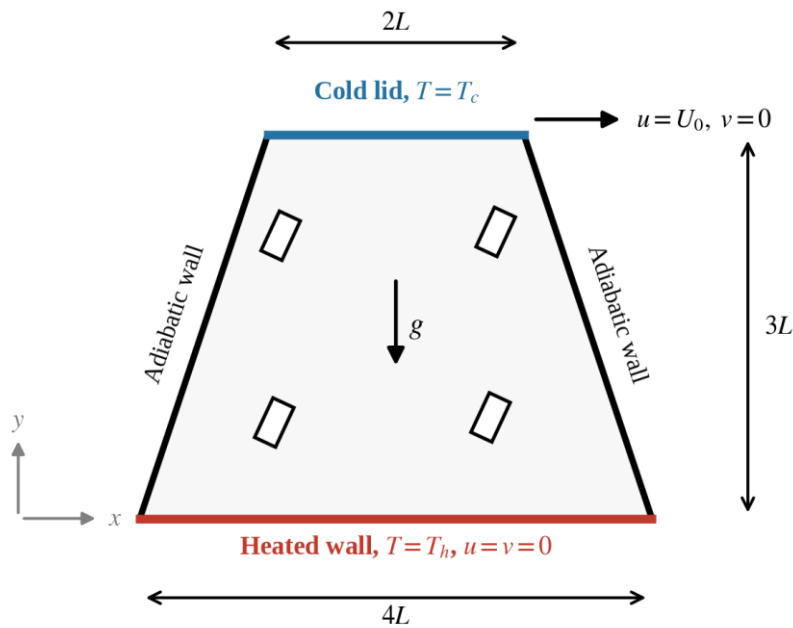
Within the studies reviewed here, none addresses a trapezoidal lid-driven cavity with a heated base, a cold moving lid, and several inclined rectangular obstacles over a wide range of Grashof and Reynolds numbers. This paper addresses that combination.

Mathematical Formulation

Physical Model and Governing Equations

Figure 1 shows the cavity: a trapezoid with a base of $4L$, a lid of $2L$, and a height of $3L$. The lid is cold and moves horizontally at constant speed U_0 ; the base is heated; the inclined side walls are adiabatic; gravity acts in the negative y direction. Four inclined rectangular obstacles are placed symmetrically, two in the upper and two in the lower half of the cavity; their surfaces are no-slip walls. The fluid is incompressible and Newtonian with constant properties except for the density in the buoyancy term (Boussinesq approximation). The flow is steady, laminar, and two-dimensional, and viscous dissipation and radiation are neglected. The Prandtl number was held constant in all simulations.

Figure 1 Schematic of the Trapezoidal Cavity With Four Inclined Obstacles



Note. Obstacle positions and sizes are schematic and not to scale.

The governing equations for continuity, momentum, and energy are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g\beta(T - T_c) \quad (3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (4)$$

where u and v are the velocity components, p is the pressure, T is the temperature, ν is the kinematic viscosity, α is the thermal diffusivity, β is the thermal expansion coefficient, g is the gravitational acceleration, and ρ is the density.

The boundary conditions are:

Lid: $u = U_0, v = 0, T = T_c$

Heated base: $u = v = 0, T = T_h$

Inclined side walls: $u = v = 0, \partial T / \partial n = 0$ (adiabatic)

Obstacle surfaces: $u = v = 0$ (no slip)

Dimensionless Form

With L as the length scale and U_0 as the velocity scale, the dimensionless variables are

$$X = \frac{x}{L}, Y = \frac{y}{L}, U = \frac{u}{U_0}, V = \frac{v}{U_0}, P = \frac{p}{\rho U_0^2}, \theta = \frac{T - T_c}{T_h - T_c} \quad (5)$$

and the equations become

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (6)$$

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{1}{\text{Re}} \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) \quad (7)$$

$$U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial Y} + \frac{1}{\text{Re}} \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \frac{\text{Gr}}{\text{Re}^2} \theta \quad (8)$$

$$U \frac{\partial \theta}{\partial X} + V \frac{\partial \theta}{\partial Y} = \frac{1}{\text{RePr}} \left(\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} \right) \quad (9)$$

$$\text{Re} = \frac{U_0 L}{\nu}, \quad \text{Gr} = \frac{g \beta (T_h - T_c) L^3}{\nu^2}, \quad \text{Pr} = \frac{\nu}{\alpha}, \quad \text{Ri} = \frac{\text{Gr}}{\text{Re}^2} \quad (10)$$

The Reynolds number Re compares inertial and viscous forces, the Grashof number Gr compares buoyancy and viscous forces, and the Prandtl number Pr compares momentum and thermal diffusion. In the cases examined here $\text{Ri} \geq 1,000$ (Table 2), so buoyancy forcing is large relative to the lid-driven inertia throughout.

Numerical Method

The pressure is eliminated with the penalty formulation (Hughes et al., 1979; Reddy & Gartling, 2010), in which the continuity equation is relaxed to

$$P = -\gamma \left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \right) \quad (11)$$

with a large penalty parameter γ . The velocity components and temperature are approximated as $U \approx \Sigma U_k \Phi_k$, $V \approx \Sigma V_k \Phi_k$, and $\theta \approx \Sigma \theta_k \Phi_k$, where

Φ_k are the shape functions. Multiplying Eqs. (7)–(9) by the weight functions Φ_i , integrating over the domain Ω , and applying the divergence theorem gives the Galerkin residuals

$$C(f) = U \frac{\partial f}{\partial X} + V \frac{\partial f}{\partial Y}, \quad D = \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \quad (12)$$

$$R_i^{(1)} = \int_{\Omega} \left[\Phi_i C(U) + \gamma \frac{\partial \Phi_i}{\partial X} D + \frac{1}{\text{Re}} \nabla \Phi_i \cdot \nabla U \right] d\Omega = 0 \quad (13)$$

$$R_i^{(2)} = \int_{\Omega} \left[\Phi_i C(V) + \gamma \frac{\partial \Phi_i}{\partial Y} D + \frac{1}{\text{Re}} \nabla \Phi_i \cdot \nabla V - \text{Ri} \Phi_i \theta \right] d\Omega = 0 \quad (14)$$

$$R_i^{(3)} = \int_{\Omega} \left[\Phi_i C(\theta) + \frac{1}{\text{RePr}} \nabla \Phi_i \cdot \nabla \theta \right] d\Omega = 0 \quad (15)$$

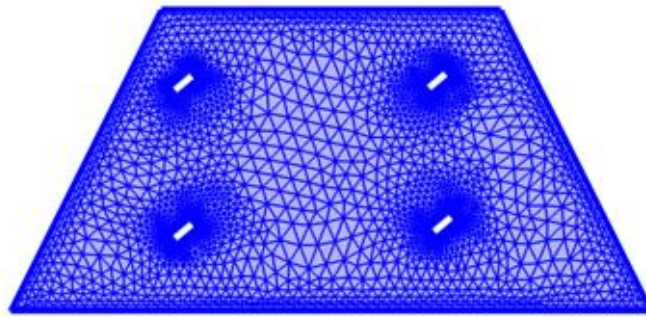
where boundary integrals have been omitted for brevity. The residuals are assembled into a global nonlinear algebraic system, which was solved in COMSOL Multiphysics (COMSOL AB, n.d.). The domain was discretised with an unstructured mesh refined around the obstacles and near the walls (Figure 2; Table 1).

Results and Discussion

Computational Mesh

Table 1 and Figure 2 describe the mesh, which contains 5,208 elements, mostly triangular, with local refinement around the four obstacles.

Figure 2 Computational Mesh



Note. Obstacles appear as white holes in the mesh.

Table 1 Mesh Statistics

Quantity	Value
Total domain elements	5,208
Triangular elements	4,760
Quadrilateral elements	448
Mesh vertices	2,953
Boundary (edge) elements	256
Vertex elements	20
Total mesh area	1.032
Minimum-to-maximum element area ratio	0.004292

Note. Values are as reported by COMSOL Multiphysics for the mesh used in all cases.

Simulation Cases

Two series of simulations were run (Table 2). In Series A, Gr was increased from 3,000 to 500,000 at $Re = 1$. In Series B, Re was increased

from 0.1 to 1.0 at $Gr = 1,000$. The peak speed in Table 2 is the maximum of the dimensionless velocity magnitude, read from the colour bar of each streamline plot.

Table 2 Simulation Cases and Peak Dimensionless Speed

Series	Case	Gr	Re	$Ri = Gr/Re^2$	Peak speed
A	1	3,000	1.0	3,000	2.19
A	2	10,000	1.0	10,000	12.79
A	3	50,000	1.0	50,000	65.46
A	4	500,000	1.0	500,000	254.7
B	5	1,000	0.1	100,000	104.37
B	6	1,000	0.3	11,111	6.15
B	7	1,000	0.7	2,041	1.36
B	8	1,000	1.0	1,000	1.17

Note. Speed is scaled by the lid velocity U_0 . Ri values are rounded.

Effect of the Grashof Number

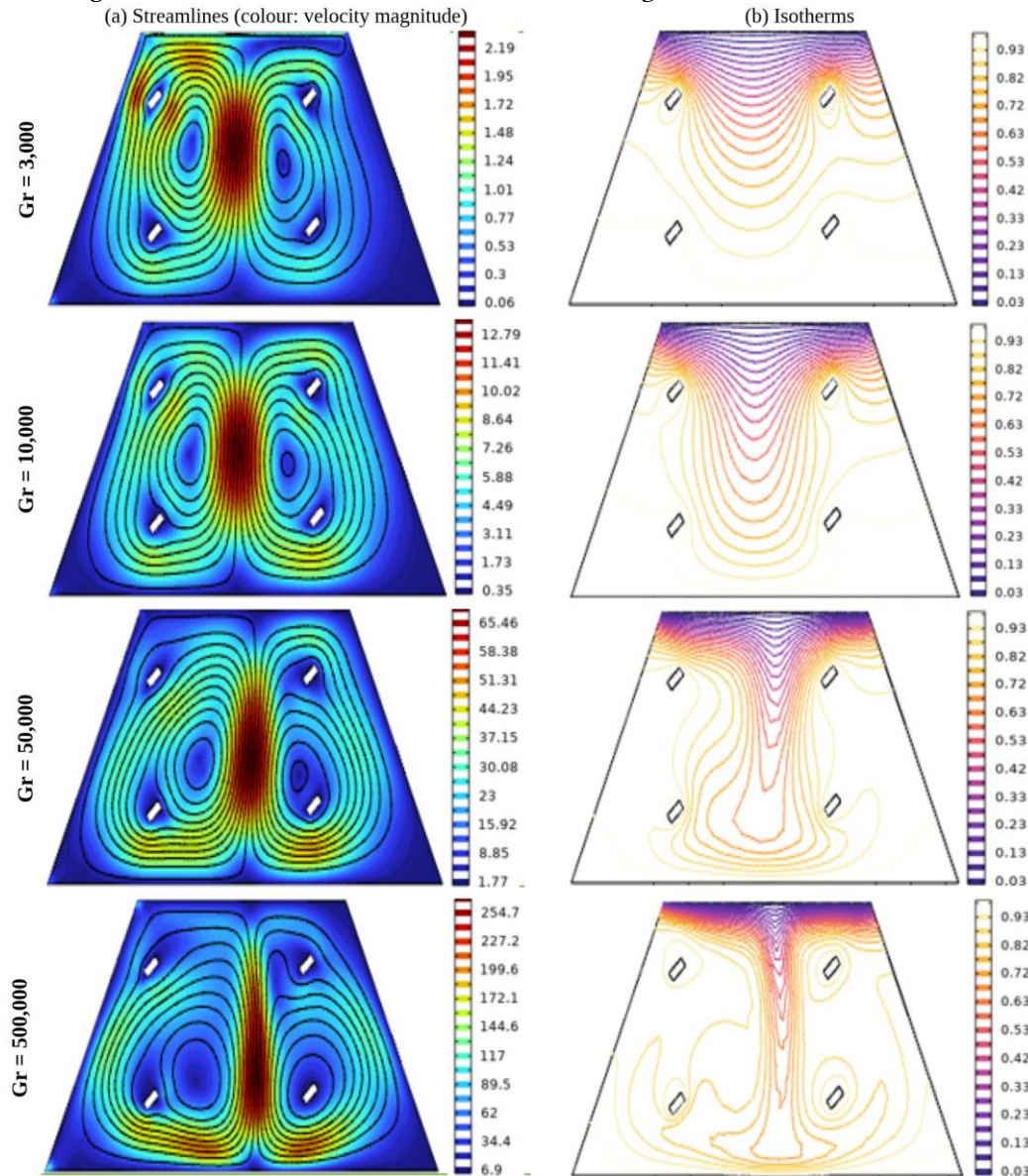
Figure 3 shows the response at $Re = 1$. At $Gr = 3,000$ the flow already consists of two nearly symmetric counter-rotating cells on either side of

a broad central jet (peak speed 2.19). The isotherms are crowded against the cold lid and bow downward in a broad bell shape in the central region, while in the lower cavity they are

widely spaced. At $Gr = 10,000$ (peak 12.79) the cells elongate, the central jet strengthens, and the isotherm bell deepens. At $Gr = 50,000$ (peak 65.46) the jet narrows into a column, streamlines crowd along the central jet and the isotherms form a narrow

central finger of cold fluid that reaches towards the base, with additional lobes beside the lower obstacles. At $Gr = 500,000$ (peak 254.7) the jet is thinner still, the thermal layer beneath the lid is very thin, and the finger extends close to the heated base.

Figure 3 Streamlines and Isotherms for Increasing Grashof Number at $Re = 1$



Note. Colour bars in (a) give the dimensionless velocity magnitude; colour bars in (b) give the dimensionless temperature θ . Results are from the authors' COMSOL simulations.

Peak speed rose by factors of 5.8, 5.1, and 3.9 for successive increases in Gr of 3.3, 5, and 10 times, so the local power-law exponent fell from 1.47 to 1.01 to 0.59. A fall towards 0.5 is what would be expected as the circulation approaches the buoyancy velocity scale, which in units of U_0 is $\sqrt{Gr/Re}$, but this interpretation rests on only

four points and was not tested against further Gr values.

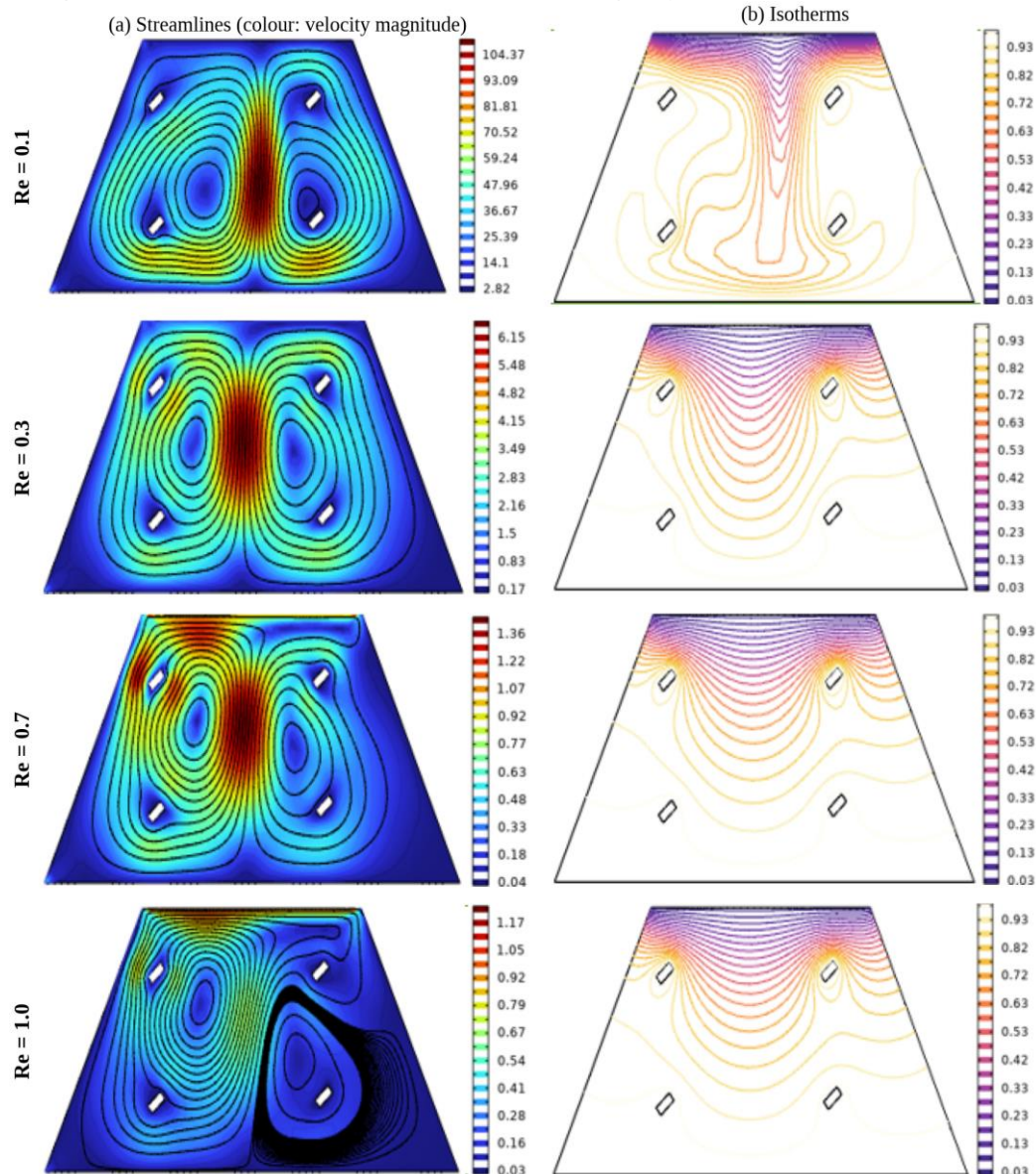
Effect of the Reynolds Number

Figure 4 shows the response at $Gr = 1,000$. Because velocities are scaled by the lid speed, a slower lid at fixed Gr means a larger Ri , so the

lowest- Re case is the most buoyancy-dominated. At $Re = 0.1$ ($Ri = 100,000$) the peak speed is 104.37, the central jet is narrow, and the isotherms form a deep central finger with lobes beside the lower obstacles, a pattern that closely resembles the $Gr = 50,000$ case in Figure 3. At $Re = 0.3$ ($Ri \approx 11,000$; peak 6.15) the pattern is a broad bell of isotherms, like the $Gr = 10,000$

case. At $Re = 0.7$ (peak 1.36) and $Re = 1.0$ (peak 1.17) the symmetry is lost: the left cell enlarges and accelerates along the lid, and at $Re = 1.0$ a separate, weaker cell is confined to the lower right while the isotherms are smooth and shallow. Peak speed therefore fell about ninety-fold as Re rose from 0.1 to 1.0.

Figure 4 *Streamlines and Isotherms for Increasing Reynolds Number at $Gr = 1,000$*



Note. Colour bars as in Figure 3. Results are from the authors' COMSOL simulations.

Morphology follows Ri closely, since cases with similar Ri gave similar patterns, but peak speed does not collapse on Ri alone (6.15 at $Ri \approx 11,000$ against 12.79 at $Ri = 10,000$), because Re

also sets the diffusion coefficients $1/Re$ and $1/(Re Pr)$ in Eqs. (7)–(9).

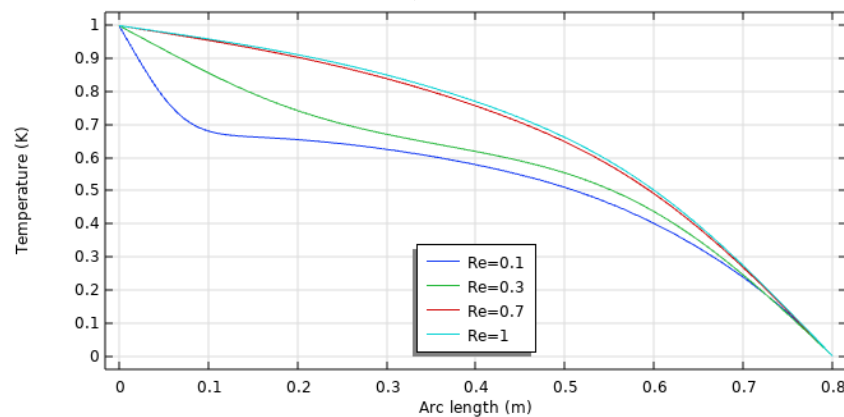
Temperature Along the Reference Line

Figure 5 plots the dimensionless temperature along a reference line from the heated wall ($\theta =$

1) to the cold wall ($\theta = 0$), with arc length 0–0.8. At $Re = 0.1$ the temperature falls quickly to about 0.67 within the first 0.1 of arc length, remains nearly flat through the core, and drops steeply near the cold end, which is the signature of a

well-mixed core. The $Re = 0.7$ and $Re = 1.0$ profiles are almost identical and decline smoothly, with a temperature of about 0.76 at arc length 0.4 against about 0.58 for $Re = 0.1$; the $Re = 0.3$ profile lies between them.

Figure 5 *Temperature Along the Reference Line From the Heated Wall to the Cold Wall at $Gr = 1,000$*



Note. The vertical axis is dimensionless temperature θ .

Table 3 lists the reported temperature gradient for each Re . It rises by 144.5% from $Re = 0.1$ to $Re = 1.0$, but nearly all of the rise occurs early (+92.6% from 0.1 to 0.3, +25.2% from 0.3 to 0.7, and +1.4% from 0.7 to 1.0), so the profile steepens and then saturates. The values are descriptive indicators of how sharply the temperature changes between the heated and cold walls, not heat transfer rates or Nusselt numbers, and they are consistent with the steeper mid-line slopes of the high- Re curves in Figure 5.

Table 3 *Reported Temperature Gradient at $Gr = 1,000$*

Re	Temperature gradient	Change from previous Re (%)
0.1	0.31554	—
0.3	0.60761	+92.6
0.7	0.76101	+25.2
1.0	0.77140	+1.4

Note. Gradient values are dimensionless; percentage changes were calculated from the tabulated values.

Interpretation in the Light of Previous Research

The strengthening of circulation and thinning of thermal layers as Gr rises is the expected behaviour of buoyancy-driven flow (Bejan, 2013). The formulation used here follows the penalty finite element approach applied to bottom-heated lid-driven cavities by Basak et al. (2009). The asymmetric states at $Re = 0.7$ and 1.0 may reflect the multiplicity of solutions reported for lid-driven trapezoidal enclosures (Bhattacharya et al., 2013); this possibility was not tested here. Compared with the obstacle-bearing cavities cited above (Shah et al., 2021;

Colak et al., 2021), the present results add the response of a four-obstacle trapezoidal cavity over a wide Ri range, although without an obstacle-free reference case the specific influence of the obstacles cannot be isolated.

Limitations

Several limitations should be noted. No grid-independence study or validation against a benchmark such as the lid-driven cavity of Ghia et al. (1982) was reported, so the numerical accuracy is unverified. The results are graphical and qualitative: peak speeds were read from colour bars, and neither the local nor the average

Nusselt number was computed. All cases include the obstacles, so their individual effect is not separated. The steady laminar assumption may not hold at the highest Gr , where unsteadiness is possible, and a single Prandtl number, one obstacle arrangement, and one cavity geometry were examined.

Conclusions and Future Work

A penalty Galerkin finite element model of laminar mixed convection in a lid-driven trapezoidal cavity with four inclined obstacles showed two counter-rotating cells flanking a central jet in all eight cases. Raising Gr strengthened the circulation by two orders of magnitude and pushed a finger of cold fluid from the lid towards the heated base. Raising Re at fixed Gr weakened the dimensionless circulation, produced smoother and more stratified isotherms, broke the symmetry of the flow, and increased the reported temperature gradient with diminishing returns. The Richardson number organised the flow morphology throughout.

Future work should add a grid-independence study and benchmark validation, compute local and average Nusselt numbers, include an obstacle-free reference cavity, vary the size, position, and thermal condition of the obstacles, test other Prandtl numbers, and examine possible unsteadiness and solution multiplicity at high Gr . Extensions to nanofluids, non-Newtonian fluids, magnetic fields, and three-dimensional geometries are also natural next steps.

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